

Output Feedback Two-Time Scale Control of Multilink Flexible Arms

B. Siciliano

Dipartimento di Informatica e Sistemistica,
Università degli Studi di Napoli Federico II,
80125 Napoli, Italy

J. V. R. Prasad

A. J. Calise

School of Aerospace Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332-0150

Lightweight flexible arms will most likely constitute the next generation robots. The design key is the adoption of flexible links, rather than rigid links like in today's industrial robots. Despite all the potential advantages achievable with a flexible arm, the control problem is complex, due to the introduction of increasingly more complex dynamics. This paper represents an effort toward the goal of designing efficient control systems for multilink flexible arms. A two-time scale approach is pursued which allows the adoption of a composite control strategy. First a slow control can be designed for the slow (rigid) subsystem, then a fast stabilizing control for the fast (flexible) subsystem. The main contribution of the paper is to address the problem of lack of full state measurements concerned with the fast control design. An output feedback dynamic compensator of fixed order is designed. Its optimal gains are computed according to a loop transfer recovery technique in order to obtain a robust design. The control is tested by means of simulation results for a nonlinear model of a two-link flexible arm.

Introduction

Lightweight flexible arms have lately been receiving the attention of an increasing number of researchers. Potentially, they may improve on the performance of conventional massive rigid industrial manipulators (Book, 1985). The main problem for modeling and controlling a flexible arm is induced by the structural flexibility (Cetinkunt and Book, 1990).

Several approaches have been proposed in the literature for modeling lightweight arms. One common denominator is the adoption of the Lagrangian technique which yields closed-form expressions of all dynamic terms. As for the inherently distributed nature of the flexible system, finite-dimensional models are needed which approximate the exact infinite-dimensional models. The recursive formulation proposed by Book (1984), suitably combined with the assumed-modes method (Meirovitch, 1967), leads to a number of generalized coordinates to handle for control purposes. Nevertheless, other methods have been studied, including finite elements (Sunada and Dubowsky, 1983; Usoro et al., 1986), polynomial expansions (Tomei and Tornambè, 1988), Hamiltonian principle (Everett, 1989; Serna and Bayo, 1989), with different implications on control design complexity.

On the other hand, it was stressed by Book et al. (1985) that the control problem for a flexible arm may be solved by combination of the following approaches:

- Design material and shapes such that high stiffness/mass ratio is achieved.
- Passive damping treatment of flexible elements to help damping out the vibrations.
- Active feedback control of flexible vibrations.

- Trajectory planning algorithms to constrain excitement of the flexible modes.

Research to understand the advantages and limitations offered by the above complementary efforts is a crucial component of making lighter robot arms desirable in practical manipulation tasks.

A variety of contributions aimed at designing control systems performing active feedback control of flexible vibrations have been proposed, employing linear control (Cannon and Schmitz, 1984; Fukuda, 1985; Hastings and Book, 1985; Sakawa et al., 1985; Chalhoub and Ulsoy, 1987; Oakley and Cannon, 1988), frequency domain techniques (Book and Majette 1985; Chait et al., 1988; Bayo et al., 1989; Yurkovich et al., 1989), adaptive control (Meldrum and Balas, 1985; Yuan et al., 1989), robust control (Korolov and Chen, 1989; Yuan and Book, 1987), transfer function approaches (Wang and Vidyasagar, 1989), inverse dynamics techniques (Asada et al., 1987; Bayo, et al., 1988), nonlinear decoupling controllers (Singh and Schy, 1986; Chedmail et al., 1987; De Luca 1989; De Luca and Siciliano, 1989; Pfeiffer, 1989), pseudolinearization methods (Nicosia et al., 1989). One limitation of most studies is that the flexible one-link case is investigated. This is much too simple a case to capture the coupling effects between rigid body and flexible body dynamics.

The approach pursued here is based on a two-time scale model of the flexible arm, as derived by Siciliano and Book (1988). A similar approach was later proposed by Khorrami and Özgüner (1988). This allows the definition of a slow subsystem corresponding to the rigid body motion, and a fast subsystem describing the flexible motion. A composite control strategy (Kokotovic, 1984) is then applied. First a slow control is designed for the slow subsystem as it would be done for an equivalent rigid arm, then a fast control stabilizes the fast subsystem.

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The main contribution of this work is to address the problem of the lack of full state availability, as the rates of the flexible variables cannot be directly measured. Previous work can be found in (Siciliano et al., 1989) for the simple case of a one-link flexible arm. Later, Calise et al., (1990) have considered a more general problem wherein the measurements are a mixture of slow and fast variables; this is the case when the rigid and first flexible mode of the one-link arm are treated together as part of the slow subsystem. The present paper extends the approach in (Siciliano et al., 1989) to the case of multilink arms and is based on the preliminary results in (Siciliano et al., 1991): A fixed order dynamic compensator for the fast subsystem is designed as in (Kramer and Calise, 1988) and a convergent numerical algorithm for calculating linear-quadratic optimal output feedback gains (Moerder and Calise, 1985) is used. Also, a loop transfer recovery formulation (Calise and Prasad, 1989) is shown to lead to a straightforward design procedure, with excellent robustness properties. An example is finally worked out for a two-link flexible arm, where the importance of nonlinear coupling terms is greatly magnified. Simulation results are presented which validate the theoretical conclusions.

2 Two-Time Scale Model of Multilink Flexible Arms

The Lagrangian-assumed modes method established by Book (1984) constitutes an effective means for deriving a dynamic model for a multi-degree-of-freedom flexible arm. In synthesis, once the kinematic description of the structure is set up, the process of obtaining the equations of motion is as follows:

- Pick the generalized coordinates to obtain a finite-dimensional model; the natural choice is the joint variables and the modal coordinates for every flexible member.
- Form the kinetic energy, the potential energy, and the virtual work expressions for the system.
- Take the necessary derivatives of the Lagrange equations and assemble the resulting equations.

The algebraic complexity of the intermediate steps which lead to the final model, according to the above procedure, represents the major drawback of the method. To this purpose, symbolic manipulation programs are an effective computational tool for modeling systems of large order, and for checking and manipulating the elements of the dynamic equations in explicit form. A method to symbolically derive the full nonlinear dynamic equations of motion for multilink flexible manipulators was proposed by Cetinkunt and Book (1989).

As a result of the above method, the equations of motion for an n -link flexible arm can be written in the closed-form

$$\mathbf{M}(\mathbf{q}, \mathbf{d}) \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{d}} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_1(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathbf{f}_2(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_1(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{d}, \dot{\mathbf{d}}) \\ \mathbf{g}_2(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{d}, \dot{\mathbf{d}}) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{Kd} \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ \mathbf{0} \end{bmatrix} \quad (1)$$

where $\mathbf{q} = (q_1 \dots q_n)^T$ is the vector of joint variable, $\mathbf{M}$ is the inertia matrix, $\mathbf{f}_1$ and $\mathbf{f}_2$ are the vectors containing gravitational, Coriolis, and centrifugal terms, $\mathbf{g}_1$ and $\mathbf{g}_2$ are the vectors accounting for the interaction of joint variables and their rates with deflection variables and their rates, $\mathbf{K} = \text{diag}(k_{11}, \dots, k_{1,m_1}, \dots, k_{n,m_1}, \dots, k_{n,m_n})$ is the positive diagonal matrix of the constant flexural stiffness coefficients, $\mathbf{u} = (u_1 \dots u_n)^T$ is the vector of input torques applied at the n joints. Two null vectors of appropriate dimensions appear in (1). As for the vector of deflection variables $\mathbf{d} = (d_{11} \dots d_{1,m_1} \dots d_{n,m_1} \dots d_{n,m_n})^T$, this is obtained as the result of an assumed modes expansion for each link—under the assumption of small deflections—truncated at m_i , $i = 1, \dots, n$ (Meirovitch, 1967). Further, it has implicitly been assumed that each link is clamped at the corresponding joint so that there will be no modal displacements at joint locations and then zero input torques. This point will be further discussed in the case study.

Since the inertia matrix $\mathbf{M}$ is symmetric and positive-definite, it can be inverted and denoted by $\mathbf{H}$, which can be partitioned as follows:

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} [n \times n] & \mathbf{H}_{12} [n \times m'] \\ \mathbf{H}_{21} [m' \times n] & \mathbf{H}_{22} [m' \times m'] \end{bmatrix} \quad (2)$$

where $m' = m_1 + \dots + m_n$ is the total number of modes. Equations (1) then become—dropping the dependences for notation compactness—

$$\ddot{\mathbf{q}} = -\mathbf{H}_{11}\mathbf{f}_1 - \mathbf{H}_{12}\mathbf{f}_2 - \mathbf{H}_{11}\mathbf{g}_1 - \mathbf{H}_{12}\mathbf{g}_2 - \mathbf{H}_{12}\mathbf{Kd} + \mathbf{H}_{11}\mathbf{u}, \quad (3a)$$

$$\ddot{\mathbf{d}} = -\mathbf{H}_{21}\mathbf{f}_1 - \mathbf{H}_{22}\mathbf{f}_2 - \mathbf{H}_{21}\mathbf{g}_1 - \mathbf{H}_{22}\mathbf{g}_2 - \mathbf{H}_{22}\mathbf{Kd} + \mathbf{H}_{21}\mathbf{u}. \quad (3b)$$

The system (3) is characterized by having $n + m'$ generalized coordinates but only n control inputs. This poses serious drawbacks for control design purposes, compared to the case of rigid arms.

A viable strategy is represented by a singular perturbation approach (Siciliano and Book, 1988), according to which a two-time scale description of the system is obtained. Specifically, the smallest stiffness constant of $\mathbf{K}$ in (1), say k_{11} , can be regarded as the inverse of a perturbation parameter, i.e., $\mu = 1/k_{11}$. For a given manipulator geometry, the limit $\mu \rightarrow 0$ corresponds to the case of an equivalent rigid manipulator. It should be stressed that the approach is a candidate to work only if μ is actually small (typically $\mu \ll 1$); this will be the case of the example discussed in Section 4. In general, however, the entries of $\mathbf{K}$ depend on the types of expansion functions used when generating the mathematical model and then the two-time scale approach is effective only when the above condition is satisfied. It is worth mentioning from a practical standpoint that it may not always be possible to separate rigid body dynamics from flexible dynamics. This depends on the location of the first flexible mode, and the bandwidth required in the slow subsystem design.

At this point, factoring $\mathbf{K}$ as $\mathbf{K} = (1/\mu)\tilde{\mathbf{K}}$ and defining the new variables (elastic forces)

$$\mathbf{z} := \frac{1}{\mu} \tilde{\mathbf{K}} \mathbf{d} \quad (4)$$

yields the equations of the system in two-time scale form

$$\ddot{\mathbf{q}} = -\mathbf{H}_{11}\mathbf{f}_1 - \mathbf{H}_{12}\mathbf{f}_2 - \mathbf{H}_{11}\mathbf{g}_1 - \mathbf{H}_{12}\mathbf{g}_2 - \mathbf{H}_{12}\mathbf{z} + \mathbf{H}_{11}\mathbf{u}, \quad (5a)$$

$$\mu\ddot{\mathbf{z}} = -\mathbf{H}_{21}\mathbf{f}_1 - \mathbf{H}_{22}\mathbf{f}_2 - \mathbf{H}_{21}\mathbf{g}_1 - \mathbf{H}_{22}\mathbf{g}_2 - \mathbf{H}_{22}\mathbf{z} + \mathbf{H}_{21}\mathbf{u} \quad (5b)$$

where the superscript “ $\sim$ ” in (5b) indicates that the corresponding quantities have been premultiplied by $\tilde{\mathbf{K}}$.

At this point, the typical steps of a singular perturbation formulation can be taken (Kokotovic, 1984). Because of the presence of μ , the system described by Eqs. (5) exhibits a boundary layer phenomenon in the fast variables $\mathbf{z}$. Formally setting $\mu = 0$ accomplishes a model order reduction from $n + m'$ to n . It can be shown (Siciliano and Book, 1988) that $\mathbf{g}_1(\mu = 0) = \mathbf{0}$ and $\mathbf{g}_2(\mu = 0) = \mathbf{0}$. Also, the $\mathbf{H}_{21}$ and $\mathbf{H}_{22}$ dependence on $\mathbf{z}$ vanishes when μ is set to 0. The differential equations (5b) degenerate into the algebraic transcendental equations

$$\mathbf{0} = -\mathbf{H}'_{21,s}\mathbf{f}_{1,s} - \mathbf{H}'_{22,s}\mathbf{f}_{2,s} - \mathbf{H}'_{22,s}\mathbf{z}_s + \mathbf{H}'_{21,s}\mathbf{u}_s \quad (6)$$

where the subscript “ s ” indicates that the corresponding quantities are computed in the slow time scale, i.e., for $\mu = 0$. Since $\mathbf{H}'_{22,s}$ is positive definite, it is possible to solve Eqs. (6) for $\mathbf{z}_s$ as

$$\mathbf{z}_s = \mathbf{H}'_{22,s}(-\mathbf{H}'_{21,s}\mathbf{f}_{1,s} + \mathbf{H}'_{21,s}\mathbf{u}_s) - \mathbf{f}_{2,s}. \quad (7)$$

Using this solution in (5a) formally yields the slow subsystem

$$\ddot{\mathbf{q}}_s = (\mathbf{H}_{11,s} - \mathbf{H}_{12,s}\mathbf{H}'_{22,s}^{-1}\mathbf{H}'_{21,s})(-\mathbf{f}_{1,s} + \mathbf{u}_s). \quad (8)$$

It can easily be checked that

$$\mathbf{H}_{11,s} - \mathbf{H}_{12,s}\mathbf{H}'_{22,s}^{-1}\mathbf{H}'_{21,s} = \mathbf{M}_{11,s}^{-1} \quad (9)$$

where $\mathbf{M}_{11,s}$ is the $(n \times n)$ positive definite matrix appearing in the model of the equivalent rigid arm.

To derive the fast (boundary-layer) subsystem, the slow variables are treated as constants. Defining the fast variables $\mathbf{z}_f = \mathbf{z} - \mathbf{z}_s$, and the fast control $\mathbf{u}_f = \mathbf{u} - \mathbf{u}_s$, the fast subsystem of (5b) becomes

$$\frac{d^2 \mathbf{z}_f}{d\tau^2} = -\mathbf{H}_{22,s} \mathbf{z}_f + \mathbf{H}_{21,s} \mathbf{u}_f \quad (10)$$

where $\tau = t/\sqrt{\mu}$ is the fast time scale.

3 Design of a Composite Controller With Fast Output Feedback

On the basis of the above two-time scale model, the design of a feedback controller for the system (5) can be performed according to a composite strategy (Kokotovic 1984); namely,

$$\mathbf{u} = \mathbf{u}_s(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{u}_f \left(\mathbf{z}_f, \frac{d\mathbf{z}_f}{d\tau} \right) \quad (11)$$

with the constraint that $\mathbf{u}_f(0,0) = \mathbf{0}$, so that $\mathbf{u}_f$ is inactive along the trajectories specified by (7).

The slow control for the slow subsystem (8) can be designed according to the well-known computed-torque concept used for rigid manipulators,

$$\mathbf{u}_s = \hat{\mathbf{M}}_{11,s} \mathbf{v} + \dots \hat{\mathbf{M}}_{11,s} \mathbf{v} + \hat{\mathbf{f}}_{1,s} \quad (12)$$

where $\hat{\mathbf{M}}_{11,s}$ and $\hat{\mathbf{f}}_{1,s}$ denote the available estimates of $\mathbf{M}_{11,s}$ and $\mathbf{f}_{1,s}$, respectively, and $\mathbf{v}$ is a new control input. This can be chosen as

$$\mathbf{v} = \ddot{\mathbf{q}}_d + \mathbf{K}_D \dot{\mathbf{e}} + \mathbf{K}_P \mathbf{e} \quad (13a)$$

where $\mathbf{q}_d$ is the desired joint trajectory, $\mathbf{e} = \mathbf{q}_d - \mathbf{q}$, and $\mathbf{K}_P$, $\mathbf{K}_D$ are suitable positive diagonal matrices which, under the assumption of perfect compensation ($\hat{\mathbf{M}}_{11,s} = \mathbf{M}_{11,s}$, $\hat{\mathbf{f}}_{1,s} = \mathbf{f}_{1,s}$) shape the response of the linear stable system

$$\ddot{\mathbf{e}} + \mathbf{K}_D \dot{\mathbf{e}} + \mathbf{K}_P \mathbf{e} = \mathbf{0}. \quad (13b)$$

At this point, it is required that the fast subsystem (10) be uniformly stable along the trajectories $\mathbf{z}_s$ given by (7). To this purpose, let a state space representation of (10) be

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}_f \quad (14a)$$

$$\mathbf{A}_{[2m' \times 2m']} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{H}_{22,s}' & \mathbf{0} \end{bmatrix} \mathbf{B}_{[2m' \times n]} = \begin{bmatrix} \mathbf{0} \\ \mathbf{H}_{21,s}' \end{bmatrix} \quad (14b)$$

where $\mathbf{x}^T = [\mathbf{z}_f^T (d\mathbf{z}_f/d\tau)^T]$, and zero block matrices of appropriate dimensions appear. The system described by equations (14) is a linear system with m' couples of poles on the imaginary axis in the s -plane. Since the pair $(\mathbf{A}, \mathbf{B})$ can be shown to be completely controllable, a fast state feedback control of the type

$$\mathbf{u}_f = \mathbf{K}_{Pf} \mathbf{z}_f + \mathbf{K}_{Df} \frac{d\mathbf{z}_f}{d\tau} \quad (15)$$

can be devised to arbitrarily place the poles of the system (14).

Under the control (11), with $\mathbf{u}_s$ as in (12) and $\mathbf{u}_f$ as in (15), the goal of tracking the desired joint trajectory together with stabilizing the deflections around the equilibrium trajectory, naturally established by the rigid system under the slow control, is achieved by an order of $\sqrt{\mu}$ approximation. A rigorous robustness analysis relative to the magnitude of the perturbation parameter and/or to practically imperfect model compensation would be in order. For instance, when the perturbation parameter is not small enough, the use of integral manifolds to obtain a more accurate slow subsystem that accounts for the effects of flexibility up to a certain order of $\sqrt{\mu}$ (Siciliano et al., 1986) represents a possible strategy to make the composite control perform satisfactorily.

On the other hand, it should be recalled that even in the

case of perfect compensation, the model is an approximate finite-dimensional model of an inherently distributed parameter system; from the control viewpoint, it is advisable to include as many modes as necessary to accurately describe the flexible dynamics of the system but to exclude those modes whose natural frequencies fall outside the bandwidth of the actuators.

Simple inspection of the fast controller in (15) reveals that full state availability is required for its practical implementation. Hastings and Book (1985) showed that the deflection variables $\mathbf{d}$ can be accurately reconstructed from strain gage measurements, whereas their rates $\dot{\mathbf{d}}$ cannot. In order to overcome this drawback, in the following an output feedback controller for the fast subsystem (14) is developed.

It is well known that LQR synthesis methods have guaranteed stability margins. Unfortunately, this property holds only in the case of full state feedback. Observer based compensator design techniques exist to estimate the unavailable plant states, and make the LQR design feasible. However, this combination of state estimation and regulation may result in a compensator design with poor stability margins, even though the separate designs are robust. It was shown by Doyle and Stein (1981) that the loop transfer properties of an LQR design can be recovered in an observer based compensator design when the plant is square and non-minimum phase. However, the order of the resulting compensator when designed for large order systems may prove unwarranted.

Optimal output design of fixed-order compensators, introduced in the early seventies by Johnson and Athans (1970) has received limited attention due to numerous difficulties associated with the design approach. Part of the difficulty lies in the fact that the compensator representation initially proposed was overparameterized. This invariably leads to convergence problems when computing an optimal solution. The other major objections are that there are no guarantees on stability margins, and there are few guidelines for selecting the distribution of initial conditions, and for penalizing the plant states to improve either performance or robustness.

More recently, Kramer and Calise (1988) showed two canonical forms for the compensator representation which provide a minimal parameterization. These representations exclude the use of direct feedthrough of the output, since direct feedback is undesirable both from the point of view of sensor noise reduction and robustness. Also, a loop transfer recovery procedure for fixed-order compensators was proposed by Calise and Prasad (1989) which uniquely defines the state and compensator weighting matrix, and the initial state distribution matrix. Below we adopt this formulation for the design of the output fast feedback controller.

The output equation for the fast subsystem (14) can be simply expressed as

$$\mathbf{y} = \mathbf{C}\mathbf{x} \quad (16a)$$

$$\mathbf{C}_{[m' \times 2m']} = [\mathbf{I} \quad \mathbf{0}] \quad (16b)$$

where the identity matrix $\mathbf{I}$ and the null matrix $\mathbf{0}$ account for the fact the variables $\mathbf{z}_f$ are available while their rates $d\mathbf{z}_f/d\tau$ are not. In other words, we assume that a number of m' strain gages are available which allow the measurements of m' link deflections $\mathbf{d}$ corresponding to the m' fast variables $\mathbf{z}_f$. We point out here that $\mathbf{y}$ is dependent only on fast states, whereas in the more general case it would depend on both slow and fast states (Calise et al., 1990).

A compensator of fixed order p , without direct feedthrough of the output, can be formulated in observer canonical form as

$$\dot{\mathbf{u}}_f = -\mathbf{H}^0 \mathbf{w} \quad (17a)$$

$$\dot{\mathbf{w}} = \mathbf{P}^0 \mathbf{w} + \mathbf{u}_p \quad (17b)$$

$$\mathbf{u}_p = \mathbf{P}_w \mathbf{u}_f - \mathbf{N}_y \quad (17c)$$

where

$$\mathbf{H}_{[n \times p]}^0 = \text{block diag}[(0, \dots, 0, 1)_{l_i \times 1}], \quad i = 1, \dots, n \quad (18)$$

and

$$\mathbf{P}_{[p \times p]}^0 = \text{block diag}[\mathbf{P}_1^0, \dots, \mathbf{P}_n^0] \quad (19a)$$

$$\mathbf{P}_{i, [l_i \times l_i]}^0 = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (19b)$$

with the observability indices l_i satisfying the following conditions: $\sum_{i=1}^n l_i = p$, and $l_i < l_{i+1}$, $i = 1, \dots, n$. In (17c), $\mathbf{N}$ and $\mathbf{P}_w$ are, respectively, a $(p \times m')$ matrix and a $(p \times n)$ matrix of free parameters.

Defining $\tilde{\mathbf{x}}^T = (\mathbf{x}^T \mathbf{w}^T)$, $\tilde{\mathbf{y}}^T = (\mathbf{y}^T - \mathbf{u}_f^T)$, and $\tilde{\mathbf{u}} = \mathbf{u}_p$, the dynamic compensator design can be expressed in terms of a standard output feedback problem for the augmented system

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\tilde{\mathbf{u}} \quad (20a)$$

$$\tilde{\mathbf{y}} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} \quad (20b)$$

$$\tilde{\mathbf{u}} = -\mathbf{G}\tilde{\mathbf{y}} \quad (20c)$$

$$\tilde{\mathbf{A}}_{[(2m' + p) \times (2m' + p)]} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{H}^0 \\ \mathbf{0} & \mathbf{P}^0 \end{bmatrix} \quad \tilde{\mathbf{B}}_{[(2m' + p) \times p]} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \quad (20d)$$

$$\tilde{\mathbf{C}}_{[(m' + n) \times (2m' + p)]} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^0 \end{bmatrix} \quad \mathbf{G}_{[p \times (m' + n)]} = [\mathbf{N} \quad \mathbf{P}_w] \quad (20e)$$

where the number of free parameters is minimized and no zero elements in $\mathbf{G}$ appear.

The output feedback problem is then optimized according to the following augmented quadratic performance index

$$J = E_{\tilde{\mathbf{x}}_0} \left[\int_0^\infty (\tilde{\mathbf{x}}^T \mathbf{Q} \tilde{\mathbf{x}} + \tilde{\mathbf{u}}^T \mathbf{R} \tilde{\mathbf{u}}) dt \right] \quad (21)$$

where $\mathbf{Q} \geq \mathbf{0}$ and $\mathbf{R} > \mathbf{0}$, and the expectation is taken over some initial distribution on $\tilde{\mathbf{x}}_0$.

The necessary conditions for optimality require the solution of the triple $(\mathbf{G}, \mathbf{S}, \mathbf{L})$ satisfying:

$$\mathbf{A}_c^T \mathbf{S} + \mathbf{S} \mathbf{A}_c + \mathbf{Q} + \tilde{\mathbf{C}}^T \mathbf{G}^T \mathbf{R} \mathbf{G} \tilde{\mathbf{C}} = \mathbf{0} \quad (22a)$$

$$\mathbf{A}_c \mathbf{L} + \mathbf{L} \mathbf{A}_c^T + \tilde{\mathbf{X}}_0 = \mathbf{0} \quad (22b)$$

$$\mathbf{R} \mathbf{G} \tilde{\mathbf{C}} \mathbf{L}^T - \tilde{\mathbf{B}}^T \mathbf{S} \mathbf{L} \tilde{\mathbf{C}}^T = \mathbf{0} \quad (22c)$$

for a stable closed-loop system matrix

$$\mathbf{A}_c = \tilde{\mathbf{A}} - \tilde{\mathbf{B}} \mathbf{G} \tilde{\mathbf{C}}. \quad (23)$$

In (22b), $\tilde{\mathbf{X}}_0 = E[\tilde{\mathbf{x}}_0 \tilde{\mathbf{x}}_0^T]$ is the variance matrix associated with the distribution assumed for the initial conditions. The convergent algorithm proposed by Moerder and Calise (1985) can be employed for solving (22a–22c). We remark that the earlier assumption of having m' strain gages below Eq. (16b) can be relaxed as long as there exists a $\mathbf{G}$ that stabilizes $\mathbf{A}_c$ in Eq. (23).

Full state feedback design is often used as a first step in designing an output feedback controller. If the fixed order compensator is designed to approximate the loop transfer properties of the full state design, then the closed-loop system should contain a set of eigenvalues and eigenvectors that approximate those of the full state design. The return vector in the case of full state design is $-\mathbf{K}^* \mathbf{x}$, where $\mathbf{K}^*$ is the optimal gain matrix specified as in (15), i.e. $\mathbf{K}^* = [\mathbf{K}_{Pf} \quad \mathbf{K}_{Df}]$. Referring to (17a), the return vector in the case of fixed-order compensator design is $-\mathbf{H}^0 \mathbf{w}$. Thus, as in (Calise and Prasad, 1989),

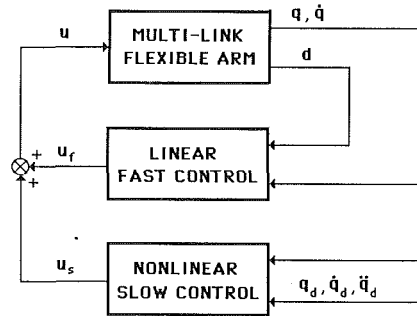


Fig. 1 Block diagram of two-time scale controller implementation

the objective in designing the compensator should be to minimize.

$$\mathbf{y}_1 = \mathbf{K}^* \mathbf{x} - \mathbf{H}^0 \mathbf{w} \quad (24)$$

for a suitably chosen input and for zero initial conditions. This naturally leads to selecting the following index of performance:

$$J = E_{\tilde{\mathbf{x}}_0} \left[\int_0^\infty (\mathbf{y}_1^T \mathbf{y}_1 + \tilde{\mathbf{u}}^T \tilde{\mathbf{u}}) dt \right]. \quad (25)$$

Substituting for $\mathbf{y}_1$ from (24) and rewriting (25) in the form of (21) leads to the following expressions for the weighting matrices:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{K}^* \mathbf{K}^{*T} & -\mathbf{K}^* \mathbf{H}^0 \\ -\mathbf{H}^0 \mathbf{K}^{*T} & \mathbf{H}^0 \mathbf{H}^0 \end{bmatrix} \quad \mathbf{R} = \rho \mathbf{I}. \quad (26)$$

Selecting the input waveforms as impulses with magnitudes uniformly distributed on the unit sphere results in the following expression for $\tilde{\mathbf{X}}_0$:

$$\tilde{\mathbf{X}}_0 = \begin{bmatrix} \mathbf{B} \mathbf{B}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (27)$$

Equations (26) and (27) uniquely define the structure of the weighting matrices needed for the fixed-order compensator design. Notice that, unlike the design of a full-order observer, the design of a fixed-order compensator depends on the gain matrix from the full state design. Moreover, this gain matrix is not implemented as a part of the final controller.

The overall two-time scale controller is to be implemented according to the block diagram of Fig. 1. In practice, a digital implementation of the analog controller derived here could be performed with different sampling rates appropriate for their respective time scales.

4 A Case Study

A numerical case study is developed to illustrate the effectiveness of the design procedure outlined in the previous section. We have chosen a planar two-link flexible arm with two assumed modes for each link. A sketch of the arm with kinematic frame assignments is shown in Fig. 2. The physical parameters of the arm with uniform mass link density are:

- link lengths: 1 m
- link masses: 1 m
- 2nd-link center of mass: 0.5 m
- 2nd-joint hub mass: 0.2 kg
- payload mass: 0.1 kg
- link inertias: 0.066 kg m²
- hub inertias: 0.1 kg m²
- payload inertia: 0.0005 kg m²
- link flexural rigidities: 1 N m²

It is worth noticing that the choice of the noninertial body-fixed rotating frames X_i, Y_i as in Fig. 2 allows the adoption of clamped boundary conditions which in turn imply zero slopes for the deflections at the joints. However, using clamped

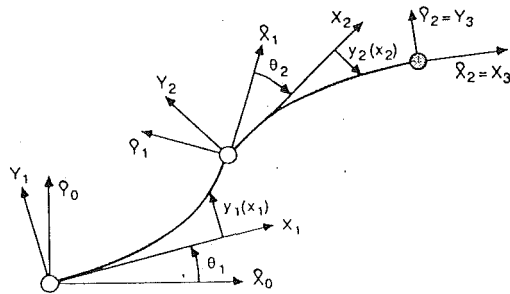


Fig. 2 The planar two-link flexible arm

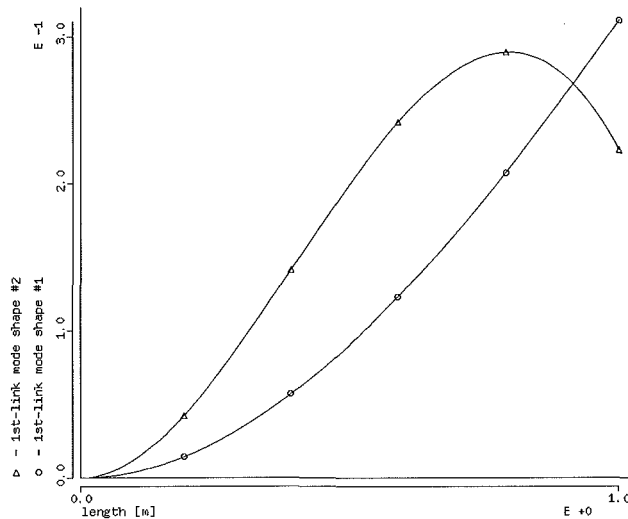


Fig. 3 Spatial eigenfunctions for the first link

modes as expansion functions is an approximation; the angles θ_1 and θ_2 are not the true joint angles, but the sum of the joint angles and the slope of the elastic motion at the joints. Therefore, the clamped modes are just trial eigenfunctions and do not represent the physical modes.

Alternatively, pinned boundary conditions can be considered for the collocated end of each link; the work of (Barbieri and Özgüner, 1988) indicated that the real behavior of the flexible slewing link is in between the two approaches, depending on the hub to beam inertia ratio. Nonetheless, it has recently been shown (Bellezza et al., 1990) that a different choice of the rotating frame—relative to pinned boundary conditions, for instance—leads to a dynamic model that is equivalent to the previous one and a proper change of coordinates exists which permits passing from one model to the other.

On the other hand, the mass boundary conditions for the non-collocated end of the inner link are difficult to satisfy since the tip mass and inertia vary with the motion of the outer link

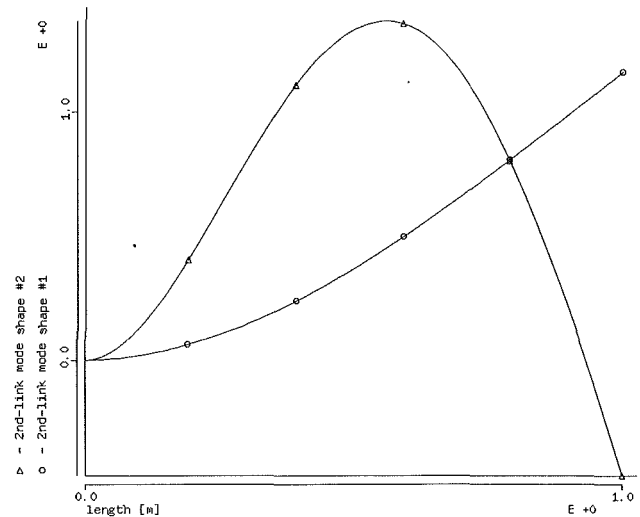


Fig. 4 Spatial eigenfunctions for the second link

the two-time scale separation. A composite control is designed first, which is composed of a slow control as in (12, 13) and a fast full state feedback control as in (15). Perfect compensation is assumed in (12), and the feedback matrices in (13) are chosen as $\mathbf{K}_P = 2.25 \mathbf{I}$ and $\mathbf{K}_D = 3\mathbf{I}$, corresponding to a double pole at -1.5 for the error dynamics at both joints. Then, a full state feedback design is carried out to stabilize the two fast modes at each link. In the open-loop, the modes have zero damping with natural frequencies at 10.3, 13.6, 67.4, 85.3 rad/s. Thus, the above choice for the slow control poles preserves the time-scale separation between the slow and the fast subsystem which is crucial for the applicability of the singular perturbation approach. A standard LQR design is undertaken for the nominal configuration of the arm with $q_2 = 0$; the matrices $\mathbf{H}'_{22,s}$ and $\mathbf{H}'_{21,s}$ in (14b) take on the following values:

$$\mathbf{H}'_{22,s} = \begin{bmatrix} 3.17 \times 10^2 & -3.15 \times 10^2 & -5.30 & -2.55 \\ -4.60 \times 10^3 & 7.06 \times 10^3 & 4.83 \times 10^1 & 2.33 \times 10^1 \\ -7.17 \times 10^1 & 4.47 \times 10^1 & 1.91 \times 10^2 & 2.04 \times 10^1 \\ -2.17 \times 10^3 & 1.35 \times 10^3 & 1.28 \times 10^3 & 4.54 \times 10^3 \end{bmatrix}$$

$$\mathbf{H}'_{21,s} = \begin{bmatrix} -2.27 \times 10^1 & -3.59 \times 10^2 \\ -1.93 \times 10^2 & 7.46 \times 10^3 \\ -2.06 \times 10^{-1} & -5.40 \times 10^1 \\ -6.26 & 1.07 \times 10^3 \end{bmatrix}$$

The weighting matrices are set to diag (0,0,0,0,0.015,0,0.005,0.0005) and diag (1,1). The resulting feedback gain matrices as in (15) are:

$$\mathbf{K}_{Pf} = \begin{bmatrix} 4.13 \times 10^{-2} & -1.99 \times 10^{-2} & -9.06 \times 10^{-2} & -4.25 \times 10^{-2} \\ -4.38 \times 10^{-1} & -2.02 \times 10^{-1} & 5.29 \times 10^{-1} & 1.33 \end{bmatrix}$$

$$\mathbf{K}_{Df} = \begin{bmatrix} -1.10 \times 10^{-1} & -5.42 \times 10^{-3} & 1.31 \times 10^{-2} & 4.05 \times 10^{-4} \\ -3.85 \times 10^{-2} & 1.15 \times 10^{-2} & -7.21 \times 10^{-2} & -1.01 \times 10^{-1} \end{bmatrix}$$

and some approximations are needed. The reader is referred to (De Luca and Siciliano, 1990a, 1990b) where all the details of the model derivation and the complete model are included.

The resulting spatial eigenfunctions, obtained with clamped-mass boundary conditions, are displayed in Figs. 3 and 4.

The perturbation parameter in (4) is $\mu = 0.0503$ justifying

which achieve damping ratios of 0.23, 0.25, 0.22, 0.31 for the four modes.

Next, an output fast feedback design with fourth-order com-

¹The arm dynamics does not depend on the first joint variable (DeLuca and Siciliano, 1990b)

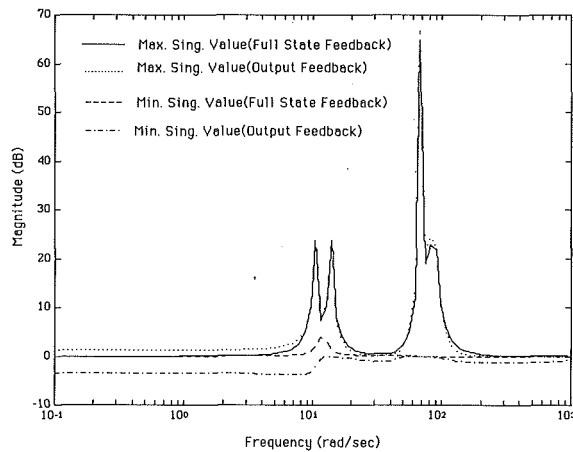


Fig. 5(a) Singular value plot of the return difference matrix with the loop broken at the input to the plant dynamics of the fast subsystem

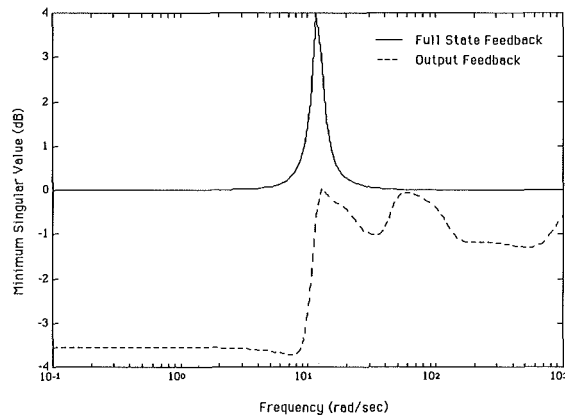


Fig. 5(b) Minimum singular value plot of the return difference matrix with the loop broken at the input to the plant dynamics of the fast subsystem

pensation— $p = 4$ in (17)—is carried out. The above full state feedback gain matrices are used to compute $\mathbf{Q}$ in (26), while $\rho = 1.0 \times 10^{-6}$. The design results in damping ratios of 0.17, 0.25, 0.25, 0.26 for the four modes, with the gain matrix in (20e) being formed by:

$$\mathbf{N} = \begin{bmatrix} 3.07 \times 10^4 & -1.19 \times 10^3 & 1.08 \times 10^3 & -1.95 \times 10^3 \\ -2.95 \times 10^3 & 1.07 \times 10^3 & -1.67 \times 10^3 & 5.89 \times 10^2 \\ 1.05 \times 10^3 & -3.87 \times 10^2 & 4.50 \times 10^2 & -4.74 \times 10^2 \\ 1.88 \times 10^4 & -6.83 \times 10^3 & 1.07 \times 10^4 & -3.84 \times 10^3 \end{bmatrix}$$

$$\mathbf{P}_w = \begin{bmatrix} -9.28 \times 10^3 & -1.04 \times 10^3 \\ 9.88 \times 10^3 & 1.42 \times 10^3 \\ -3.28 \times 10^3 & -3.97 \times 10^2 \\ -6.26 \times 10^4 & -8.94 \times 10^3 \end{bmatrix}$$

Figure (5a) illustrates the extent of recovery achieved by the output feedback design. Shown is a comparison of the singular value plots for the return difference matrix with the loop broken at the input to the plant dynamics of the fast subsystem. An expanded view of the minimum singular value plots of Fig. (5a) is shown in Fig. 5(b). For the full state feedback design, the minimum singular value over all frequencies is greater than or equal to 1.0. This ensures multivariable gain margins of -6 dB in the downward direction and ∞ in the upward direction. The guaranteed multivariable phase margins are ± 60 deg. The minimum singular value for the fourth order compensator reaches a lower bound of 0.65—0.6 is generally re-

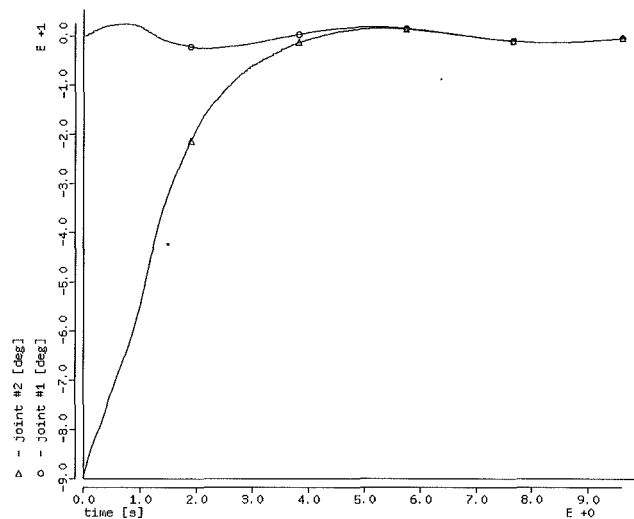


Fig. 6 Time history of joints with slow control

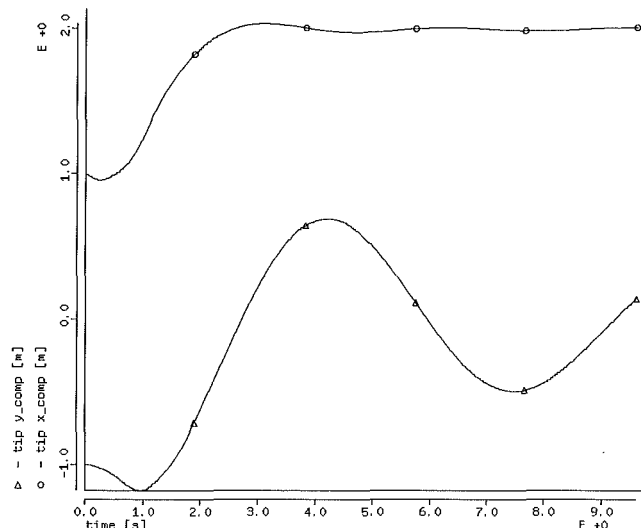


Fig. 7 Time history of tip with slow control

garded as a good design value for practical applications—at a frequency of 7.2 rad/s. This corresponds to multivariable gain margins of -4.4 dB in the downward direction and 9.2 dB in the upward direction. The multivariable phase margins are ± 38 deg.

In the following, a set of simulation is presented. A joint motion is commanded from $\mathbf{q}_d(0) = (0 - \pi/2)^T$ rad to $\mathbf{q}_d(T) = (0 \ 0)^T$ rad with the velocity profile $\dot{\mathbf{q}}_d(t) = (\pi/2)[1 + \sin(2\pi t/T - \pi/2)]$ rad/s, $0 \leq t \leq T$, with $T = 4$ s. A fifth order Runge-Kutta-Merson method has been implemented to integrate the nonlinear differential equations (1) with a sampling rate for the controller of 1.0×10^{-3} s.

Three controllers have been applied; the slow control, the composite control with full state fast feedback, and the composite control with output fast feedback.

The results in Figs. 6 and 7 indicate that with the slow control, the performance of the arm joints is acceptable while that of the tip is not; the y -component of the tip keeps vibrating about the final target position while the x -component is quite unaffected by the imposed motion. This is a direct consequence of the fact that the slow controller is unable to positively affect the vibratory motion which is greatly determined by the fast dynamics of the system. It is worth noticing, also, the typical non-minimum phase response of the tip; this is induced by the fact that one is attempting to control a non-colocated output of the system (arm tip) with a colocated input (joint torques).

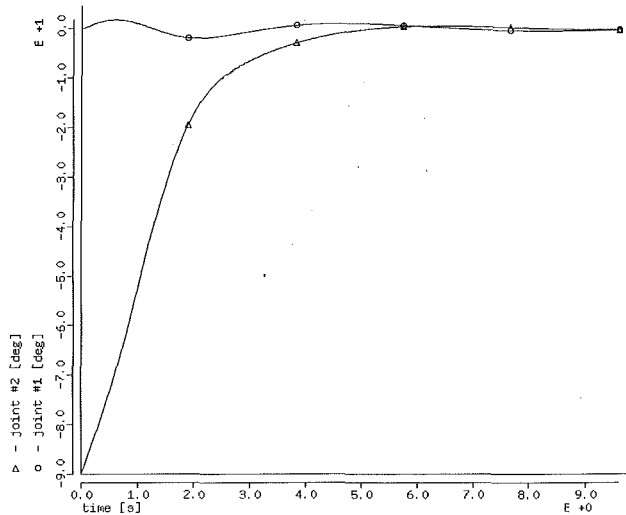


Fig. 8 Time history of joints with composite full state feedback control

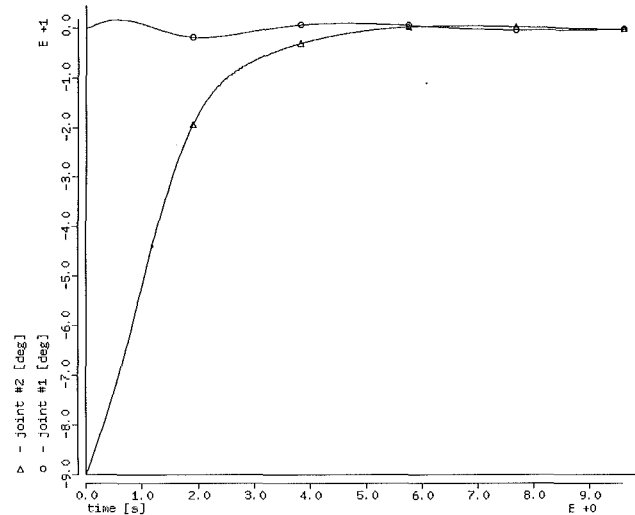


Fig. 10 Time history of joints with composite output feedback control

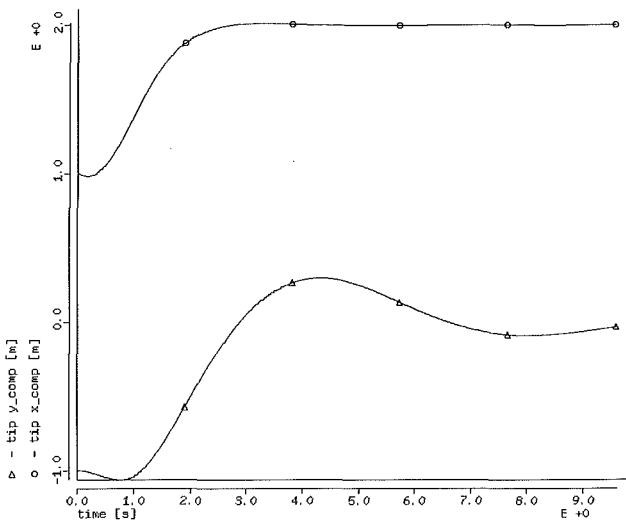


Fig. 9 Time history of tip with composite full state feedback control

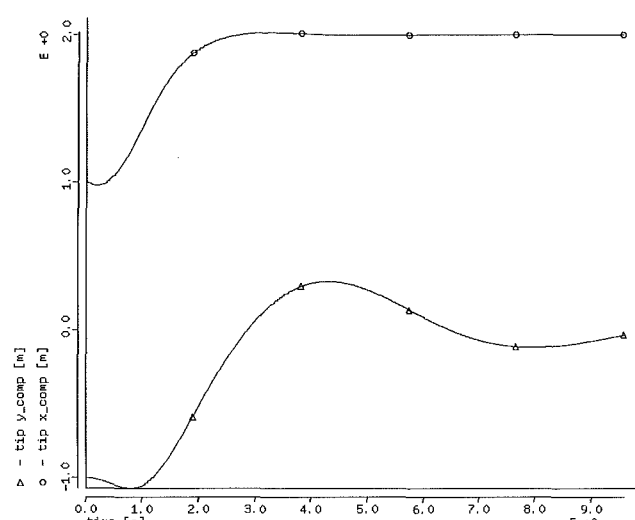


Fig. 11 Time history of tip with composite output feedback control

With the introduction of the fast control with full state feedback, the vibratory motion of the arm tip tends to die out (Fig. 9) due to the stabilizing effect of the LQR design. Correspondingly, the joint arm motion is not appreciably modified (Fig. 8) with respect to the previous case (Fig. 6) because the fast control was purposely made inactive along the trajectories of the slow subsystem as well as a good time scale separation exists in the present case.

Finally, the results in Figs. 10 and 11 demonstrate that the performance obtained by replacing the full state design with the output feedback design is still satisfactory. Therefore, the introduction of the output feedback compensator of fourth order not only ensures good recovery for the linear fast subsystem but also does not degrade the performance of the overall nonlinear system, compared with the ideal case of full state feedback. This result is of great interest for practical implementation of the proposed two-time scale control strategy.

Conclusions

A two-time scale approach has been developed for the control of multi-link flexible arms. The design of a composite control has been presented: This consists of a slow control design that can be performed as for the equivalent rigid arm, and of a fast control design for a simple linear system describing

the flexible dynamics. The problem of lack of full state measurements has successfully been solved by adopting a fixed-order dynamic compensator design combined with the loop transfer recovery technique. The simulated case study for a planar two-link flexible arm has illustrated the effectiveness of the formulation and overall design methodology in the case when a true time scale separation exists and perfect slow subsystem model compensation is assumed.

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