

Robust IDA-PBC for *Underactuated* Mechanical Systems Subject to Matched Disturbances

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Abstract—The problem of robustification of Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) for *underactuated* mechanical system *vis-à-vis* matched, constant, unknown disturbances is addressed in the paper. This is achieved adding an outer-loop controller to the IDA-PBC. Three designs are proposed, with the first one being a simple nonlinear PI, while the second and the third ones are nonlinear PIDs. While all controllers ensure stability of the desired equilibrium in spite of the presence of the disturbances, the inclusion of the derivative term allows us to inject further damping enlarging the class of systems for which asymptotic stability is ensured.

I. INTRODUCTION

Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC), first introduced in [1], is a highly popular controller design technique applicable for equilibrium stabilisation of a wide class of physical systems. A comprehensive discussion of IDA-PBC may be found in [2]. Its application for *underactuated mechanical systems* has been particularly successful as reported, for instance, in [3], [4], [5], [6]. It is widely recognised that IDA-PBC designs are robust against parameter uncertainties and unmodelled dynamics, *e.g.*, passive effects like friction. However, the (unavoidable) presence of external disturbances degrades its performance, shifting the equilibrium of the closed-loop and, possibly, inducing instability. For this reason the problem of robustification of IDA-PBC *vis-à-vis* external disturbances is of primary importance.

For *fully-actuated* mechanical systems this problem has been addressed in [7], where the key idea of adding an integral action that preserves the port-Hamiltonian (pH) structure of the system, first proposed in [8], is exploited. Proposition 5 of [7] presents a dynamic nonlinear controller, including the essential integral action,

that ensures global asymptotic stability (GAS) of the desired equilibrium in spite of the presence of constant matched disturbances. The case of unmatched and/or time-varying disturbances is also studied in that paper, where controllers that ensure input-state-stability are proposed. In [9] the particular case of *constant* mass matrix is considered. It is claimed that adding an integral action in velocities rejects the matched disturbances. Unfortunately, as indicated in [7], this claim is wrong.

To the best of the authors' knowledge, the first attempt to solve the constant, disturbance rejection problem for *underactuated* mechanical systems was published in [10]. The authors consider the simplest case of 2DOF mechanical systems with constant mass matrix and underactuation degree one. Although the main idea is interesting, there are several unfortunate errors that invalidate the claims. Indeed, it is easy to show that the proposed control law—given in equation (27) of [10]—does not satisfy the matching equations, which is a key step in the design. Moreover, since it is not possible to inject damping in all the momentum coordinates, the closed-loop damping matrix is not full-rank, a critical assumption made in [10] to claim asymptotic stability.

In this paper we complement the main idea of [10], that is adding an integral action on non-passive outputs of underactuated systems, with the developments of [7] to propose an outer-loop controller that solves the problem of (constant, matched) disturbance rejection for underactuated n -DOF mechanical systems with arbitrary underactuation degree. An interesting feature of the proposed outer-loops is that they preserve in closed-loop the port-Hamiltonian (pH) form of the mechanical system. The design is applicable for systems where the mass matrix is *independent of the unactuated* coordinates and the closed-loop mass matrix is *constant*. The first assumption is instrumental for the results reported in [3], [4], [6], where it is imposed to simplify the kinetic energy matching equation. In the present work this assumption, as well as the requirement that the closed-loop mass matrix is constant, are needed to construct a suitable change of coordinates under which the integral action is added. It should be noted that these assumptions are verified by a large class of underactuated mechanical systems, including those considered in [4].

The rest of the paper is organized as follows. Section II briefly recalls IDA-PBC and formulates its

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robustification problem. The main result of the paper is presented in Section III. Section IV contains the example of the acrobot [4]. Future work is discussed in Section V.

Notation. For $x \in \mathbb{R}^n$, $S \in \mathbb{R}^{n \times n}$, $S = S^\top > 0$, we denote $|x|^2 := x^\top x$ and $\|x\|_S^2 := x^\top Sx$. Given a function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ we define $\nabla H := \left(\frac{\partial H}{\partial x}\right)^\top$.

II. ROBUST IDA-PBC

A. Standard IDA-PBC

IDA-PBC was introduced in [5] to control underactuated mechanical systems described in pH form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & I_n \\ -I_n & 0_{n \times n} \end{bmatrix} \nabla H(q, p) + \begin{bmatrix} 0_{n \times m} \\ G(q) \end{bmatrix} u, \quad (1)$$

where $q, p \in \mathbb{R}^n$ are the generalized position and momenta, respectively, $u \in \mathbb{R}^m$ is the control, $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ with $\text{rank}(G) = m < n$, the function $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$$H(q, p) := \frac{1}{2} p^\top M^{-1}(q) p + V(q) \quad (2)$$

is the total energy with $M : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, the positive definite inertia matrix and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ the potential energy.

The control objective is to design a static, state-feedback that assigns to the closed-loop the stable equilibrium $(q, p) = (q^*, 0)$, $q^* \in \mathbb{R}^n$. This is achieved in IDA-PBC by matching the pH target dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & M^{-1}(q) M_d(q) \\ -M_d(q) M^{-1}(q) & J_2(q, p) - R_d(q) \end{bmatrix} \nabla H_d \quad (3)$$

with the new total energy function $H_d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$$H_d(q, p) := \frac{1}{2} p^\top M_d^{-1}(q) p + V_d(q), \quad (4)$$

where the desired inertia matrix $M_d : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is positive definite, the desired potential energy $V_d : \mathbb{R}^n \rightarrow \mathbb{R}$ verifies

$$q_\star = \arg \min V_d(q), \quad (5)$$

and the desired damping matrix is defined by

$$R_d(q) := G(q) K_P G^\top(q) \geq 0,$$

with $K_P \in \mathbb{R}^{n \times n}$ a free positive definite matrix. The matrix $J_2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is free to the designer and fulfills the skew-symmetry condition

$$J_2(q, p) = -J_2^\top(q, p). \quad (6)$$

The closed-loop system (3) has a stable equilibrium point at $(q^*, 0)$ with Lyapunov function H_d , which verifies

$$\dot{H}_d = -\|G^\top M_d^{-1} p\|_{K_P}^2 \leq 0.$$

The closed-loop is asymptotically stable provided that the output

$$y_d := G^\top M_d^{-1} p \quad (7)$$

is detectable [11].

Equating the right-hand sides of (1) and (3) one obtains the so-called matching equations, which are two partial differential equations (PDEs) that identify the assignable M_d and V_d , and gives an explicit expression for the (static state-feedback) control signal denoted $u_{\text{IDA}}(q, p)$.

B. Formulation of the robust IDA-PBC problem

In this paper, we consider the effect of constant, matched disturbances in the mechanical system (1) that may represent external forces or an input measurement bias. We assume that the system (1) is in closed-loop with an IDA-PBC into which the disturbance propagates and must be rejected with an outer-loop control—denoted v —as requested below.

Problem formulation. Given the dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G K_P G^\top \end{bmatrix} \nabla H_d + \begin{bmatrix} 0_{n \times m} \\ G \end{bmatrix} (v + d), \quad (8)$$

with H_d as in (4) and $d \in \mathbb{R}^m$. Find (if possible) a dynamic controller $v = \beta(q, p, \zeta)$, where $\zeta \in \mathbb{R}^m$ is the state of the controller, that ensures GAS of the desired equilibrium $(q, p, \zeta) = (q^*, 0, \zeta^*)$, for some $\zeta^* \in \mathbb{R}^m$, even under the action of constant disturbances d .

Remark 1: Notice that the dimension of the dynamic extension ζ coincides with the one of the input space, *i.e.*, m . As will be shown below, this choice suffices to provide a solution to the problem.

III. MAIN RESULT

A. The class of mechanical systems

In this section, we present the main result of the paper, which requires the following assumption imposed throughout the rest of the paper.

Assumption A. The input matrix G and the desired mass matrix M_d are constant and the mass matrix $M(q)$ is independent of the non-actuated coordinates. Consequently,

$$G^\perp \nabla_q (p^\top M^{-1} p) = 0,$$

where $G^\perp \in \mathbb{R}^{(n-m) \times n}$ is a full-rank left-annihilator of G .

Remark 2: The term $G^\perp \nabla_q (p^\top M^{-1} p)$ appears in the kinetic energy matching equation as a forcing term that makes the PDE inhomogeneous and introduces a quadratic term in the unknown M_d , rendering very difficult its solution. In [3] it is also assumed to be zero to provide an explicit solution of the PDE. In [6] changes of coordinates are introduced to eliminate, or simplify, this term. In Proposition 2 of [6] it is shown that a sufficient condition to eliminate this term is that the Coriolis and centrifugal forces of the mechanical system enter into the kernel of G .

B. Nonlinear PI controller

Proposition 1: Consider the dynamics (8) with $J_2 = 0$ in closed-loop with the PI controller $v = \beta(q, \zeta)$, with

$$\beta(q, \zeta) = -K_2 K_I K_2^\top G^\top M^{-1} \nabla V_d - K_P K_I \zeta \quad (9)$$

and

$$\dot{\zeta} = K_2^\top G^\top M^{-1} \nabla V_d, \quad (10)$$

with constant matrices $K_P > 0$, $K_I > 0$ and

$$K_2 := (G^\top M_d^{-1} G)^{-1} \geq 0.$$

P1. Introduce the globally defined change of coordinates $z = \psi(q, p, \zeta)$ given by

$$z_1 = q \quad (11)$$

$$z_2 = p + G K_2 K_I (\zeta - \alpha) \quad (12)$$

$$z_3 = \zeta \quad (13)$$

The closed-loop dynamics can be written in pH form as follows

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & M^{-1} M_d & -M^{-1} G K_2 \\ -M_d M^{-1} & -G K_P G^\top & 0_{n \times m} \\ K_2^\top G^\top M^{-1} & 0_{m \times n} & 0_{m \times m} \end{bmatrix} \nabla H_z \quad (14)$$

with Hamiltonian

$$H_z(z) = \frac{1}{2} z_2^\top M_d^{-1} z_2 + V_d(z_1) + \frac{1}{2} \|z_3 - \alpha\|_{K_I}^2 \quad (15)$$

and

$$\alpha := (K_P K_I)^{-1} d.$$

P2. The equilibrium $(q, p, \zeta) = (q^*, 0, \alpha)$ is stable.

P3. If the output

$$y_{D1} = G^\top M_d^{-1} z_2$$

is a detectable output of the dynamics (14), then $(q^*, 0, \alpha)$ is an asymptotically stable equilibrium.

Proof: First, to prove **P1**, we differentiate (11) to obtain

$$\begin{aligned} \dot{z}_1 &\equiv \dot{q}, \\ &= M^{-1} z_2 - M^{-1} G K_2 K_I (z_3 - \alpha), \end{aligned}$$

which allows us to write the first state equation of (14). We proceed in a similar fashion with (12)

$$\begin{aligned} \dot{z}_2 &= \dot{p} + G K_2 K_I \dot{\zeta} \\ &= -M_d M^{-1} \nabla V_d(q) - G K_P G^\top M_d^{-1} p + G(v + d) \\ &\quad + G K_2 K_I \dot{\zeta} \\ &= -M_d M^{-1} \nabla V_d(q) - G K_P G^\top M_d^{-1} p + G d \\ &\quad + G \left[-K_2 K_I K_2^\top G^\top M^{-1} \nabla V_d - K_P G^\top M_d^{-1} G \right. \\ &\quad \left. \times K_2 K_I \zeta \right] + G K_2 K_I K_2^\top G^\top M^{-1} \nabla V_d \Big|_{(q,p,\zeta)=\psi^{-1}(z)} \\ &\equiv -M_d M^{-1} \nabla V_d(z_1) - G K_P G^\top M_d^{-1} z_2, \end{aligned}$$

which is the second row of the closed-loop dynamics (14). Finally, we note that the dynamics of ζ and z_3 are equivalent. Indeed, from the last row of (14), we obtain

$$\begin{aligned} \dot{z}_3 &= K_2^\top G^\top M^{-1} \nabla V_d(z_1) \Big|_{z=\psi(q,p,\zeta)} \\ &\equiv \dot{\zeta}, \end{aligned}$$

To prove the stability property **P2**, we consider H_z as Lyapunov candidate function for the closed-loop (14). Its time derivative results as follows

$$\dot{H}_z = -\|G^\top M_d^{-1} z_2\|_{K_P} \leq 0, \quad (16)$$

which ensures stability of the desired equilibrium. The assumption that y_{D1} is a detectable output of the system (14) ensures asymptotic stability of the equilibrium $(q^*, 0, \alpha)$ [11], proving property **P3**. ■

Remark 3: The controller (9) is a nonlinear PI (around ∇V_d) of the form

$$\begin{aligned} v &= -\mathcal{K}_P(q) \nabla V_d - \mathcal{K}_I(q) \zeta \\ \dot{\zeta} &= \nabla V_d \end{aligned}$$

with nonlinear gains

$$\begin{aligned} \mathcal{K}_P(q) &:= K_2 K_I K_2^\top G^\top M^{-1}(q) \\ \mathcal{K}_I(q) &:= K_P K_I K_2^\top G^\top M^{-1}(q). \end{aligned}$$

Remark 4: Notice that in the simplest case when V_d is quadratic, that is,

$$V_d(q) = \|q - q^*\|_S^2,$$

with $S > 0$, then v is a standard PI around the *position error* $q - q^*$. This is the case if the system (1) is linear, whence the gains $\mathcal{K}_P$ and $\mathcal{K}_I$ are constant.

C. First nonlinear PID controller

Now, we extend the previous proposition by adding damping in the coordinates z_1 to relax the detectability condition needed to ensure asymptotic stability and simplify its analysis.

Proposition 2: Consider the dynamics (8) with $J_2 = 0$ in closed-loop with the PID controller $v = \beta(q, \zeta, p)$, with

$$\begin{aligned} \beta(q, \zeta, p) &= -\left(K_P G^\top M_d^{-1} G K_1 G^\top M^{-1} + K_1 G^\top \dot{M}^{-1} \right) \\ &\quad \times \nabla V_d - K_1 G^\top M^{-1} \nabla^2 V_d M^{-1} p - K_I \zeta \end{aligned} \quad (17)$$

and¹

$$\dot{\zeta} = G^\top M_d^{-1} G K_1 G^\top M^{-1} \nabla V_d + G^\top M_d^{-1} p, \quad (18)$$

with constant matrices $K_P > 0$, $K_I > 0$ and $K_1 > 0$.

P1. The closed-loop dynamics can be written in pH form as follows

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 & M^{-1} M_d & 0_{n \times m} \\ -M_d M^{-1} & -G K_P G^\top & -G \\ 0_{m \times n} & G^\top & 0_{m \times m} \end{bmatrix} \nabla H_z \quad (19)$$

with

$$\Gamma_1 := M^{-1} G K_1 G^\top M^{-1} \geq 0,$$

¹To simplify the notation we have used $\dot{M}^{-1}$ to denote $\frac{d(M^{-1})}{dt}$.

Hamiltonian as in (15) and $\alpha := K_I^{-1}d$. The states of (8) and (19) are related by the state transformation $z = \psi(q, p, \zeta)$ as follows

$$z_1 = q \quad (20)$$

$$z_2 = p + GK_1 G^\top M^{-1} \nabla V_d \quad (21)$$

$$z_3 = \zeta \quad (22)$$

P2. The equilibrium $(q, p, \zeta) = (q^*, 0, \alpha)$ is stable.

P3. If the output

$$y_{D2} = \begin{bmatrix} G^\top M^{-1} \nabla V_d \\ G^\top M_d^{-1} z_2 \end{bmatrix}$$

is a detectable output of the dynamics (19), then $(q^*, 0, \alpha)$ is an asymptotically stable equilibrium of the closed-loop.

Proof: First, to prove **P1**, we take the state transformation (20), and we differentiate it with respect to time to obtain

$$\begin{aligned} \dot{z}_1 &= -\Gamma_1 \nabla V_d + M^{-1} z_2 \\ &= -\Gamma_1 \nabla V_d + M^{-1} [p + M\Gamma_1 \nabla V_d] \\ &\equiv \dot{q}, \end{aligned}$$

which allows us to write the first state equation of (14). We proceed in a similar fashion with the state transformation (21). By differentiating with respect to time, we obtain

$$\begin{aligned} \dot{z}_2 &= \dot{p} + GK_1 G^\top M^{-1} \nabla^2 V_d \dot{q} + GK_1 G^\top \dot{M}^{-1} \nabla V_d \\ &= -M_d M^{-1} \nabla V_d(q) - GK_P G^\top M_d^{-1} p + G(v + d) \\ &\quad + GK_1 G^\top M^{-1} \nabla^2 V_d M^{-1} p + GK_1 G^\top \dot{M}^{-1} \nabla V_d \\ &= -M_d M^{-1} \nabla V_d(q) - GK_P G^\top M_d^{-1} \left[p + GK_1 G^\top \right. \\ &\quad \left. \times M^{-1} \nabla V_d \right] - GK_I (\zeta - \alpha) \Big|_{(q,p,\zeta)=\psi^{-1}(z)} \\ &\equiv -M_d M^{-1} \nabla V_d(z_1) - GK_P G^\top M_d^{-1} z_2, \end{aligned}$$

which is the second row of the closed-loop dynamics (19). Finally, we note that the dynamics of ζ and z_3 are equivalent. Indeed, from the last row of (19), we obtain

$$\begin{aligned} \dot{z}_3 &= G^\top M_d^{-1} z_2 \Big|_{z=\psi(q,p,\zeta)} \\ &\equiv \dot{\zeta}, \end{aligned}$$

The proof of **P2** and **P3** mimics the proof of Proposition 1, noting that

$$\dot{H}_z = -\|G^\top M^{-1} \nabla V_d\|_{K_1}^2 - \|G^\top M_d^{-1} z_2\|_{K_P}^2.$$

Remark 5: The controller (17) differs from the nonlinear PI (9) in two respects. First, it contains a *derivative term* with respect to the signal ∇V_d —hence the addition of the letter D to the PI name. Second, besides the

integral action (around ∇V_d) there is another one around $\dot{q}$. Indeed, (17) can be written in the form

$$\begin{aligned} v &= -\mathcal{K}_P(q) \nabla V_d - \zeta_1 - \zeta_2 - \mathcal{K}_D(q) \frac{d \nabla V_d}{dt} \\ \dot{\zeta}_1 &= \mathcal{K}_{I1}(q) \nabla V_d \\ \dot{\zeta}_2 &= \mathcal{K}_{I2}(q) \dot{q} \end{aligned}$$

with some suitable defined nonlinear gains $\mathcal{K}_P(q)$, $\mathcal{K}_D(q)$ and $\mathcal{K}_{Ii}(q)$, $i = 1, 2$.

Remark 6: It is important to underscore the presence of the positive semidefinite matrix Γ_1 in the (1,1) block of the system matrix in (19). This additional damping term allows—via the addition of the term $G^\top M^{-1} \nabla V_d$ in y_{D2} —to relax the condition P3 for asymptotic stability.

D. Second nonlinear PID controller

We present now a more elaborated nonlinear PID controller with the following features.

- To simplify the asymptotic stability analysis it adds damping in coordinates z_3 , as well as in z_1 and z_2 .
- The assumption of $J_2 = 0$ is obviated, enlarging the class of closed-loop systems (8) to be considered.

Proposition 3: Consider the dynamics (8) in closed-loop with the PID controller $v = \beta(q, \zeta, p)$, with

$$\begin{aligned} \beta(q, \zeta, p) &= - \left[K_P G^\top M_d^{-1} G K_1 G^\top M^{-1} + K_1 G^\top \dot{M}^{-1} \right. \\ &\quad \left. + K_2 K_I (K_2^\top + K_3^\top G^\top M_d^{-1} G K_1) G^\top M^{-1} \right] \nabla V_d \\ &\quad - \left[K_1 G^\top M^{-1} \nabla^2 V_d M^{-1} + (G^\top G)^{-1} G^\top J_2 M_d^{-1} \right. \\ &\quad \left. + K_2 K_I K_3^\top G^\top M_d^{-1} \right] p \\ &\quad - (K_P G^\top M_d^{-1} G K_2 + K_3) K_I \zeta, \end{aligned} \quad (23)$$

and

$$\begin{aligned} \dot{\zeta} &= (K_2^\top + K_3^\top G^\top M_d^{-1} G K_1) G^\top M^{-1} \nabla V_d \\ &\quad + K_3^\top G^\top M_d^{-1} p, \end{aligned} \quad (24)$$

where $K_1 > 0$, $K_P > 0$, and $K_I > 0$, $K_3 > 0$ and

$$K_2 := (G^\top M_d^{-1} G)^{-1}.$$

P1. The closed-loop dynamics can be written in pH form as follows

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 & M^{-1} M_d & -\Gamma_2 \\ -M_d M^{-1} & -GK_P G^\top & -GK_3 \\ \Gamma_2^\top & K_3^\top G^\top & -\Gamma_3 \end{bmatrix} \nabla H_z \quad (25)$$

with Hamiltonian as in (15), and the constant gains

$$\begin{aligned} \Gamma_1 &:= M^{-1} G K_1 G^\top M^{-1} \\ \Gamma_2 &:= M^{-1} G K_2 \\ \Gamma_3 &:= K_3^\top G^\top M_d^{-1} G K_2 \\ \alpha &:= K_I^{-1} (K_P G^\top M_d^{-1} G K_2 + K_3)^{-1} d. \end{aligned}$$

The states of (8) and (25) are related by the state transformation $z = \psi(q, p, \zeta)$ as follows

$$z_1 = q \quad (26)$$

$$z_2 = p + GK_1 G^\top M^{-1} \nabla V_d(q) + GK_2 K_I(\zeta - \alpha) \quad (27)$$

$$z_3 = \zeta \quad (28)$$

P2. The equilibrium $(q, p, \zeta) = (q^*, 0, \alpha)$ is stable.

P3. If the output

$$y_{D3} = \begin{bmatrix} G^\top M^{-1} \nabla V_d \\ G^\top M_d^{-1} z_2 \\ K_I(z_3 - \alpha) \end{bmatrix}$$

is a detectable output of the dynamics (25), then $(q^*, 0, \alpha)$ is an asymptotically stable equilibrium of the closed-loop.

The proof of proposition 3 follows a similar procedure as in proof of proposition 2, and thus we omit it due to space limitation.

Remark 7: Note that the controller in Proposition 1 can be derived from the one in Proposition 3 setting $K_1 = 0$ and $K_3 = 0$. An extra term needs to be added to deal with the case when $J_2 \neq 0$. Similarly, the controller in Proposition 2 can be derived from the one in Proposition 3 by setting $K_2 = 0$ and $K_3 = I_m$, and including a term to handle $J_2 \neq 0$.

IV. THE ACROBOT EXAMPLE

In this section we consider the example of the Acrobot as case study [12]. The IDA-PBC controller used in this section was borrowed from [4]. It is interesting to note that all the mechanical systems studied in that paper, including the Acrobot, belong to the class of systems we consider in our work.

A. Dynamic model and IDA-PBC

The equation of motion of the acrobot are given by (1) with $n = 1$, $m = 1$,

$$\begin{aligned} M(q_2) &= \begin{bmatrix} c_1 + c_2 + 2c_3 \cos(q_2) & c_2 + c_3 \cos(q_2) \\ c_2 + c_3 \cos(q_2) & c_2 \end{bmatrix}, \\ V(q) &= g[c_4 \cos(q_1) + c_5 \cos(q_1 + q_2)], \\ G &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

where g is the gravitational constant and c_1, c_2, c_3 and c_4 are constant parameters of the system.

The upright equilibrium $q^* = (0, 0)$ of the Acrobot can be stabilised asymptotically with the IDA-PBC controller

$$\begin{aligned} u_{\text{IDA}}(q, p) &= \frac{1}{2} \nabla_{q_2} (p^\top M^{-1} p) + \nabla_{q_2} V - \begin{bmatrix} k_2 & k_3 \end{bmatrix} \times \\ &\quad \times M^{-1} V_d + \frac{k_v}{k_1 k_3 - k_2^2} (k_2 p_1 - k_1 p_2), \end{aligned}$$

where $k_v > 0$ is the damping injection gain, and the controller gains k_1, k_2 and k_3 with

$$k_1 := \left(1 - \sqrt{c_1/c_2}\right) k_2 > 0, \quad k_3 > \frac{k_2}{1 - \sqrt{c_1/c_2}}.$$

The desired mass matrix is

$$M_d = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} > 0,$$

and the gradient of desired potential energy is given by

$$\begin{aligned} \nabla_{q_1} V_d &= -K_0 \sin(q_1 - \mu q_2) - b_1 \sin(q_1) - \\ &\quad - b_2 \sin(q_1 + q_2) - b_3 \sin(q_1 + 2q_2) - \\ &\quad - b_4 \sin(q_1 - q_2) + k_u(q_1 - \mu q_2), \\ \nabla_{q_2} V_d &= K_0 \mu \sin(q_1 - \mu q_2) - b_2 \sin(q_1 + q_2) - \\ &\quad - 2b_3 \sin(q_1 + 2q_2) + b_4 \sin(q_1 - q_2) - \\ &\quad - k_u \mu(q_1 - \mu q_2). \end{aligned}$$

where $b_1 := \frac{g}{2k_2}(c_3 c_5 \pm 2\sqrt{c_1 c_2} c_4)$, $b_2 := \frac{g\mu}{2k_2(\mu+1)}(c_3 c_4 \pm 2\sqrt{c_1 c_2} c_5)$, $b_3 := \frac{g\mu c_3 c_5}{2k_2(\mu+2)}$ and $b_4 := \frac{g\mu c_3 c_4}{2k_2(\mu-1)}$.

Remark 8: It is important to underscore that the IDA-PBC given above ensures asymptotic stability of the upward Acrobot position with a domain of attraction including a region in the lower half plane, *i.e.*, $|q_1| > \frac{\pi}{2}$. That is, the IDA-PBC can swing up the Acrobot *without switching*. To the best of the authors' knowledge this is the first such result for any pendular system. See [4] for further details of the controller design and the stability proof.

B. Nonlinear PID outer-loop: Simulations

First, we observe that G and the desired mass matrix are constant and $G^\perp \nabla_q (p^\top M^{-1} p) = 0$, thus **Assumption A** is verified. Hence to reject the disturbances we add to the IDA-PBC the outer-loop controller (23), (24) of Proposition 3 with the parameters K_1, K_3, K_I and K_P to be chosen.

To assess the performance of the proposed outer-loop controller some simulations were carried out. For the simulations, we use the values of the model parameters and the gains of the IDA-PBC provided in [4], that is, $g = 9.8$, $c_1 = 2.3333$, $c_2 = 5.3333$, $c_3 = 2$, $c_4 = 3$, $c_5 = 2$, $k_1 = 0.3386$, $k_2 = 1$, $k_3 = 5.9073$, $\mu = -0.6019$, $k_0 = 10$, $k_u = 280$ and $k_v = 12$. The gains of the nonlinear PID controller (23), (24) are as follows: $K_1 = 0.01$, $K_3 = 25$, $K_I = 0.01$, and $K_P = k_v$.

The simulations are performed under the following extreme scenario: the system starts with the Acrobot hanging down with zero velocity, that is with initial conditions $q_1(0) = -\pi$, $q_2(0) = 0$, $p_1(0) = 0$ and $p_2(0) = 0$ and without any disturbance. Then, a matched constant disturbance $d = 10\text{Nm}$ is added to the system at time $t = 25\text{s}$ (see Figure 3). Under this scenario, we test the IDA-PBC with and without the nonlinear PID.

Figures 1 shows the time histories of the angles and angular velocities of the Acrobot when only the IDA-PBC is used. As expected, the presence of the disturbance produces an error on the regulation task by shifting the equilibrium of the closed loop from the desired equilibrium.

In a second test, we simulate, under the same scenario as before, the Acrobot in closed loop with the IDA-PBC

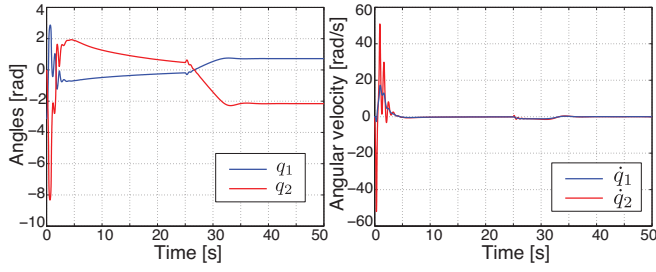


Fig. 1. Time histories of the Acrobot angles q_1 and q_2 , and the velocities $\dot{q}_1$ and $\dot{q}_2$ with the IDA-PBC only.

augmented with the nonlinear PID controller (23), (24). Figure 2 shows the time histories of the Acrobot's angles and angular velocities. It is clear, from the plots, that the angles converge to the desired position, while the velocities converge to zero. Figure 3 shows the time history of the controller state ζ , which provides the disturbance rejection. Note that the plot of $\zeta(t)$ in Figure 3 has been multiplied by the constant $a = (K_P G^T M_d^{-1} G K_2 + K_3) K_I$, such that $a\zeta(t)$ converges to $a\alpha = d$. The time history of the control input u is shown in Figure 3, that is the input produced by the IDA-PBC plus the PID controller.

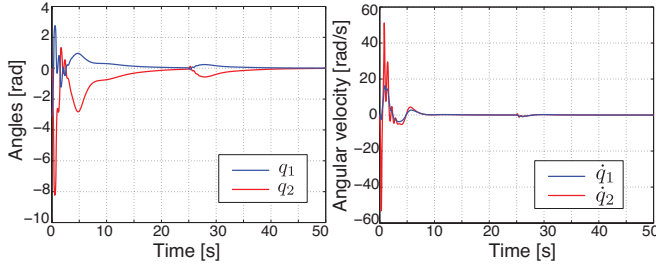


Fig. 2. Time histories of the Acrobot angles q_1 and q_2 , the velocities $\dot{q}_1$ and $\dot{q}_2$ with the IDA-PBC plus the nonlinear PID.

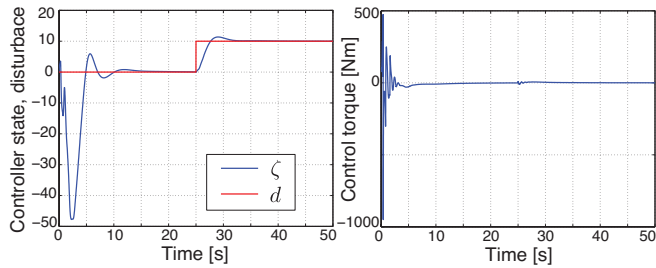


Fig. 3. Time histories of the matched disturbance d , the controller state ζ multiplied by the constant $a = (K_P G^T M_d^{-1} G K_2 + K_3) K_I$, and the control torque.

V. CONCLUSIONS

In this paper, we presented an outer-loop control design to improve the robustness of IDA-PBC for *under-actuated* mechanical system subject to matched constant

disturbances. First, it is shown that a nonlinear PI ensures stability of the desired equilibrium and, under some additional assumption, also asymptotic stability. To relax the latter assumption, enlarging the class of systems for which convergence is ensured, additional damping terms are added to the closed loop. This leads to the inclusion of a derivative term in the control law, yielding a nonlinear PID.

In future work we attempt to relax **Assumption A**, that imposes serious constraints on the class of systems for which IDA-PBC is applicable. Also, in the spirit of [7], we are considering the presence of matched and unmatched disturbances simultaneously, which might be possibly time-varying.

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