

# FOUNDATIONS OF ROBOTICS

## Motion Control

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# Outline

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- The Motion Control Problem
- Single Joint Control
- Regulation in Joint Space
- Trajectory Tracking in Joint Space
- Control in Task Space

# The Motion Control Problem

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- The control problem for a robot consists in determining suitable input commands that allow the correct (or the best possible) execution of a planned task, both in nominal conditions and in the presence of model uncertainties or external disturbances
- Controlling a robot can be interpreted in multiple ways, according to the level of definition
  - at a high level, controlling a robot could mean to successfully complete a work program
  - at an intermediate level, it implies the capability of following portions of desired trajectories
  - at a low level, it requires the joint position error to be zeroed

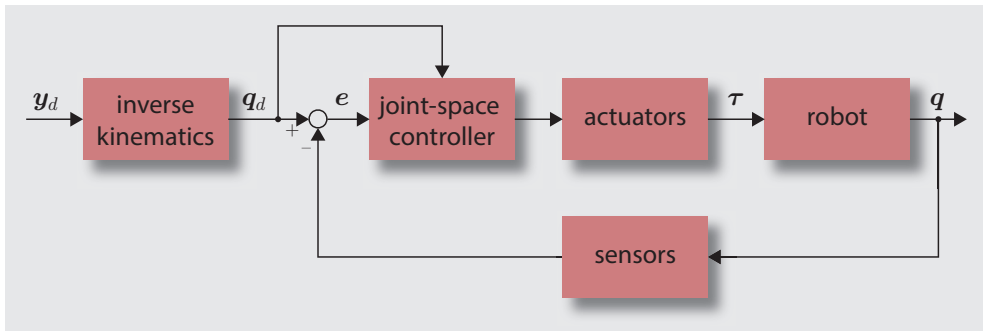
- The performance of a robot control system is evaluated from two points of view
  - nominal task execution, e.g., velocity/speed of motion completion, positioning accuracy, and energy requirements
  - robustness with respect to uncertainty or perturbations, including repeatability, parameter variation, or measurement noise
- Robustness mainly stems from the generalized application of feedback strategies, i.e., real-time use of sensor information, as well as extensive data collection and processing over repeated trials for learning purposes
- Control design is discussed under the assumption that the robot is not in contact with the environment, i.e., in the case of pure **motion control**
  - **regulation**, in which the robot must be driven to a desired setpoint in configuration space or in task space (e.g., a desired end-effector posture)
  - **trajectory tracking**, in which the robot must follow a desired trajectory in configuration space or in task space

- Robot dynamics is highly nonlinear and the motion of each link is generally coupled with the motion of the others
- A solution can only be found by a **centralized** approach which looks simultaneously at the whole robot
- Uniform performance is difficult to guarantee without resorting to some information about the dynamic model of the robot
- With motors connected to the joints by gears with large reduction ratios, each joint is approximately described by a linear and decoupled system
- This can be effectively controlled using a **decentralized** approach, treating neglected coupling terms as a disturbance to be rejected as much as possible

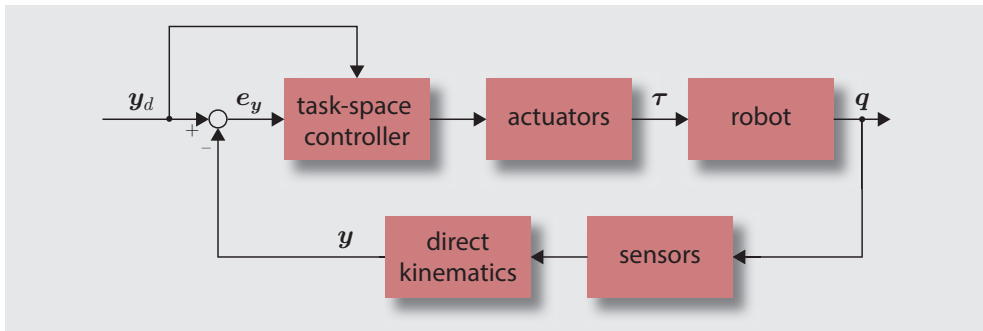
- Low-level actuator controllers are ubiquitous in all industrial robots (and beyond)
- This allows the implementation of **kinematic control**, in which purely kinematic commands are generated and sent as references to the actuator control loops
  - there are many situations in which the behavior of kinematic controllers is not satisfactory
  - this happens for highly dynamic motions, or when low-reduction gears or direct-drive actuation are adopted
- In all these cases, **dynamic control** laws are required and the actuator control design must be suitably modified to allow the execution of commands that are defined at the torque level

- The desired robot motion can be specified by a reference in joint space or in task space, denoted by  $\mathbf{q}_d$  or  $\mathbf{y}_d$
- Control actions are always defined in the joint (or actuator) space
- Two types of schemes are considered, namely, **joint-space control** and **task-space control**
- Control is obtained by the combination of a **feedback action** and a **feedforward action**
- In joint-space control, the error is  $\mathbf{e} = \mathbf{q}_d - \mathbf{q}$
- In task-space control, the error is  $\mathbf{e}_y = \mathbf{y}_d - \mathbf{y}$
- Both schemes produce a control signal that the actuators convert into a joint torque  $\boldsymbol{\tau}$

- General scheme of joint-space control



- General scheme of task-space control



- Joint-space control is typically decomposed in two subproblems
  - inverse kinematics maps  $\mathbf{y}_d$  to a corresponding desired motion  $\mathbf{q}_d$
  - a control scheme is designed so that  $\mathbf{q}$  reproduces  $\mathbf{q}_d$  and the error  $\mathbf{e}$  is zeroed asymptotically
  - with this approach, task-space variables  $\mathbf{y}$  are essentially controlled in an open-loop fashion through the robot mechanical structure
- Task-space control acts directly on the task error, embedding inverse kinematics into the feedback control loop
  - this implies an increased computational load and requires the capability to handle singularities or redundancy in real time
- Sampling intervals in the order of 0.5 to 10 ms are considered satisfactory for most robot control problems

# Single Joint Control

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- Design of low-level actuator control laws for a single robot joint including actuator dynamics
- Electrical motors are the most common actuators in robotics
- Modeling and control techniques for the case of a DC motor

## Modeling a DC Motor

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- Consider a DC motor driven by the armature voltage  $v_a$  and denote its angular position by  $\theta_m$
- Electrical equation

$$L_a \frac{di_a}{dt} = v_a - R_a i_a - k_v \omega_m$$

- $R_a$  and  $L_a$  armature resistance and inductance
    - $k_v$  back electromotive force constant
    - $\omega_m = \dot{\theta}_m$  motor angular velocity
  - Torque relation
- $$\tau_a = k_t i_a$$
- $k_t$  torque constant numerically equal to  $k_v$  in the SI unit system

- Mechanical equation

$$I_m \dot{\omega}_m + B_m \omega_m = \tau_a - \tau_l$$

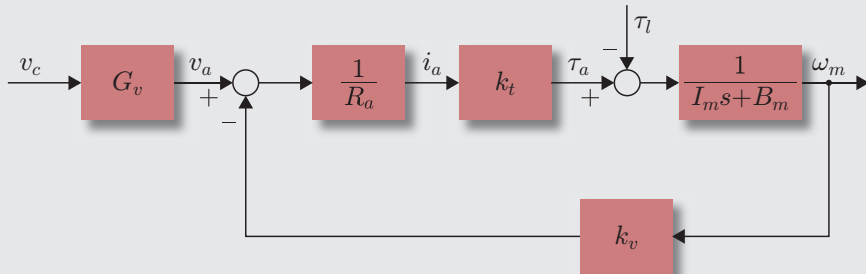
- $I_m$  motor moment of inertia
- $B_m$  viscous friction coefficient
- $\tau_l$  load torque acting as a disturbance

- In the Laplace domain

$$(L_a s + R_a) i_a(s) = v_a(s) - k_v \omega_m(s)$$

$$(I_m s + B_m) \omega_m(s) = k_t i_a(s) - \tau_l(s)$$

- DC motor block diagram



- Simplified motor model

- neglect the armature inductance dynamics and include an amplifier with gain  $G_v > 0$
- velocity response

$$\omega_m(s) = \frac{1}{I_m s + B'_m} \left( \frac{G_v k_t}{R_a} v_c(s) - \tau_l(s) \right)$$

- modified viscous term

$$B'_m = B_m + \frac{k_v k_t}{R_a}$$

- steady-state response to a constant voltage  $v_c$

$$\omega_m \approx \frac{G_v}{k_v} v_c$$

- assume  $B_m$  is negligible with respect to the other term in  $B'_m$
- in this case the motor behaves as a **velocity generator**

- DC motor with current feedback

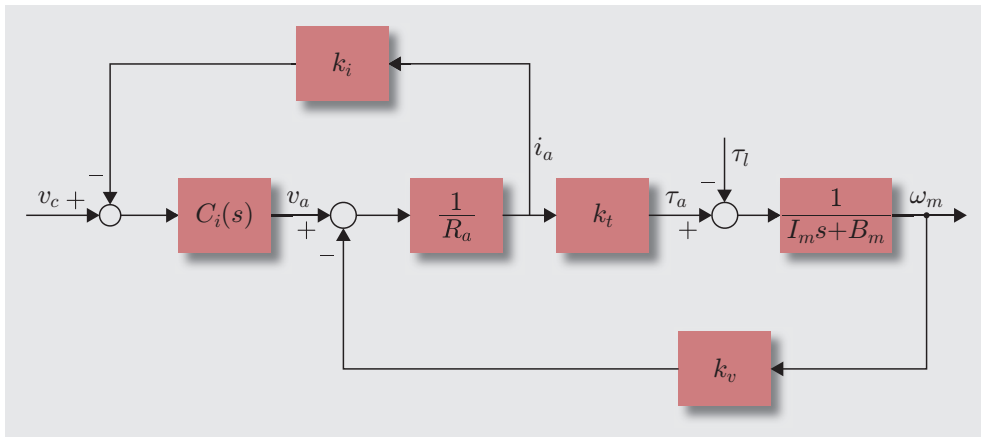
- armature current measurement via a transducer of constant  $k_i$  and a high gain controller  $C_i(s)$
- approximate torque relation

$$\tau_a = k_t i_a \approx \frac{k_t}{k_i} v_c$$

- the last approximate equality is an equality if the current controller  $C_i(s)$  contains an integral action
- in this case the motor behaves as a **torque generator**
- velocity response

$$\omega_m(s) = \frac{1}{I_m s + B_m} \left( \frac{k_t}{k_i} v_c(s) - \tau_l(s) \right)$$

- Block diagram of DC motor with current feedback



- Unified representation

$$\omega_m(s) = M(s) (\tau_m(s) - \tau_l(s))$$

- input relation

$$\tau_m = \alpha_m v_c$$

- motor transfer function

$$M(s) = \frac{G_m}{1 + T_m s}$$

- For a motor used as velocity generator

$$G_m = \frac{1}{B'_m} \quad T_m = \frac{I_m}{B'_m} \quad \alpha_m = \frac{G_v k_t}{R_a}$$

- For a motor used as torque generator

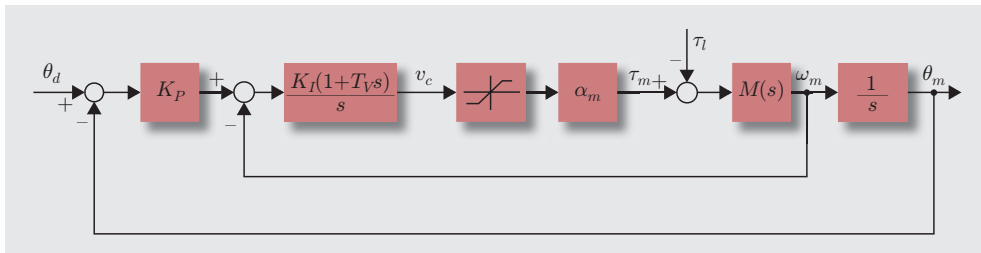
$$G_m = \frac{1}{B_m} \quad T_m = \frac{I_m}{B_m} \quad \alpha_m = \frac{k_t}{k_i}$$

## Position Control

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- Regulate the angular position  $\theta_m$  of the motor to a constant reference value  $\theta_d$
- Cascaded design with two feedback loops
  - inner velocity loop
  - outer position loop
- Measurements from a position encoder with a numerical filter for velocity reconstruction
- Nonlinear saturation represents a limitation on the motor driving voltage and on the torque that can actually be delivered

- Block scheme of servomotor for position control



- Inner velocity loop PI controller transfer function

$$\frac{K_I}{s} + K_V = \frac{K_I(1 + T_V s)}{s}$$

- $K_I > 0$
- $T_V = K_V/K_I > 0$
- integral action achieves steady-state rejection of constant load disturbances
- outer position loop uses a proportional controller with gain  $K_P$

- With  $T_V = T_m$  the closed-loop transfer function becomes

$$W(s) = \frac{\theta_m(s)}{\theta_d(s)} = \frac{1}{1 + \frac{s}{K_P} + \frac{s^2}{\alpha_m G_m K_P K_I}}$$

- Canonical second-order transfer function

$$\frac{1}{1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}}$$

- Choice of  $K_P$  and  $K_I$  for given  $\omega_n$  and  $\zeta$

$$K_P = \frac{\omega_n}{2\zeta} \quad K_I = \frac{2\zeta\omega_n}{\alpha_m G_m}$$

- Disturbance transfer function from  $\tau_l$  to  $\theta_m$

$$W_d(s) = \frac{\theta_m(s)}{\tau_l(s)} = -\frac{s}{\alpha_m K_P K_I (1 + T_m s)} \frac{1}{1 + \frac{s}{K_P} + \frac{s^2}{\alpha_m G_m K_P K_I}}$$

- The zero at the origin confirms that constant disturbances are rejected at steady state
- Output recovery time** as dominant time constant

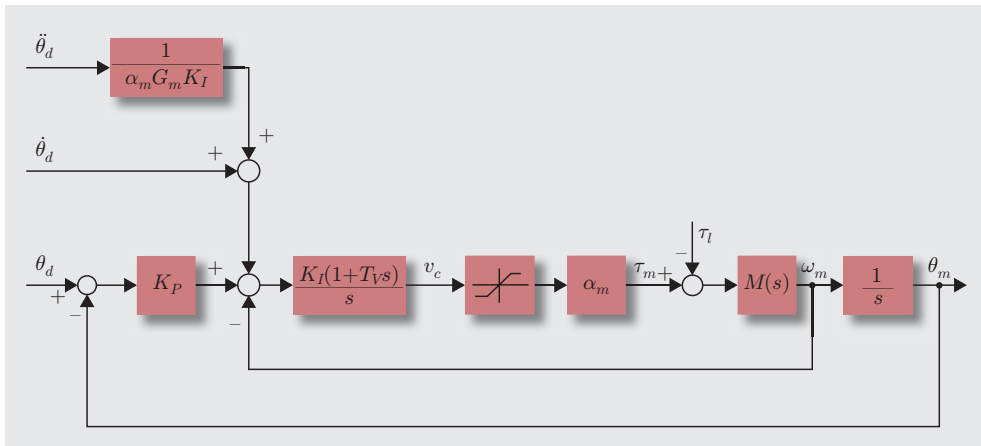
$$T_R = \max \left\{ T_m, \frac{2}{\alpha_m G_m K_I} \right\}$$

## Including a Feedforward Action

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- The cascaded scheme guarantees regulation for constant  $\theta_d$  and constant load torque
- With a time-varying reference  $\theta_d(t)$  tracking performance is degraded
- A feedforward action allows recovering error convergence

- Block diagram of control with feedforward action



- Modified reference signal

$$\theta'_d(s) = \frac{1}{W(s)} \theta_d(s) = \left( 1 + \frac{s}{K_P} + \frac{s^2}{\alpha_m G_m K_P K_I} \right) \theta_d(s)$$

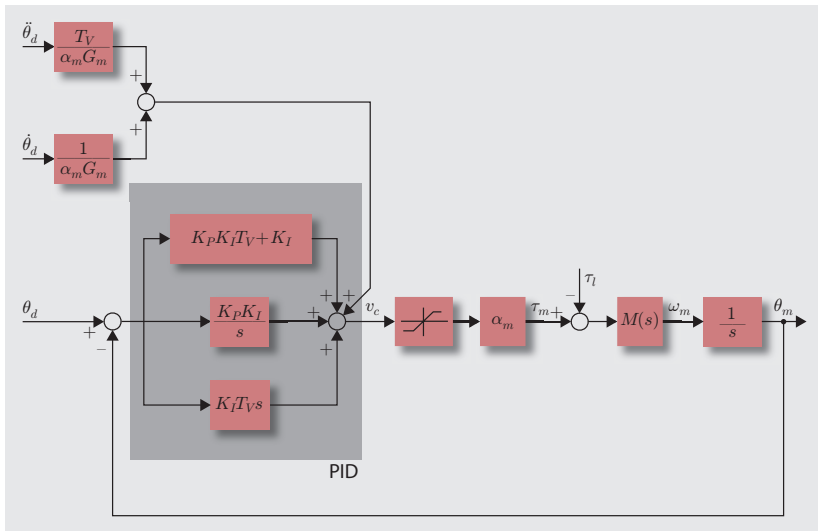
- The transfer function from  $\theta_d$  to  $\theta_m$  becomes unitary and stable
- Exact tracking at steady state for any reference time varying or not
- Desired velocity  $\dot{\theta}_d$  and acceleration  $\ddot{\theta}_d$  are computed from the desired trajectory
- Ideal performance assumes perfect knowledge of the model parameters

- Control voltage

$$\begin{aligned}
 v_c(s) &= \frac{K_I(1 + T_V s)}{s} (K_P + s)(\theta_d(s) - \theta_m(s)) + \frac{1}{\alpha_m G_m} (1 + T_V s) \dot{\theta}_d(s) \\
 &= \left( K_P K_I T_V + K_I + \frac{K_P K_I}{s} + K_I T_V s \right) (\theta_d(s) - \theta_m(s)) \\
 &\quad + \frac{1}{\alpha_m G_m} \dot{\theta}_d(s) + \frac{T_V}{\alpha_m G_m} \ddot{\theta}_d(s)
 \end{aligned}$$

- Equivalent to a proportional–integral–derivative (PID) control scheme with feedforward action
  - the PID scheme only requires position feedback

- Block diagram of equivalent PID control



# Regulation in Joint Space

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- Dynamic model in the absence of interaction and friction generalized forces

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

- Equilibrium points are all states  $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}_e, \mathbf{0})$
- Equilibria can be forced with  $\mathbf{g}(\mathbf{q}_e) \neq \mathbf{0}$
- Equilibria can be unforced with  $\mathbf{g}(\mathbf{q}_e) = \mathbf{0}$
- Regulation in joint space determines torques  $\boldsymbol{\tau}$  that drive the robot toward the desired equilibrium  $(\mathbf{q}_d, \mathbf{0})$

## Independent PID Regulation

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- The robot is treated as formed by  $n$  single-input single-output subsystems
- Each joint is controlled independently
- Inertial velocity and gravitational coupling among the joints are treated as disturbance inputs
- Motor dynamics for the  $i$ -th joint

$$I_{mi}\ddot{\theta}_{mi} + B_{mi}\dot{\theta}_{mi} = \tau_{mi} - \tau_{li}$$

- Link side dynamics

$$\tau_q = M_l(q)\ddot{q} + c(q, \dot{q}) + g(q)$$

- $i$ -th component of the dynamic model

$$\tau_{qi} = (m_{ii,a} + m_{ii,b}(q)) \ddot{q}_i + \sum_{j=1, \dots, n}^{j \neq i} m_{ij}(q) \ddot{q}_j + c_i(q, \dot{q}) + g_i(q)$$

- Gear relation between link acceleration and motor acceleration

$$\ddot{q}_i = \frac{\ddot{\theta}_{mi}}{n_{ri}}$$

- Relation between joint torque and load torque

$$\tau_{qi} = n_{ri} \tau_{li}$$

- Power equality

$$\tau_{li} \dot{\theta}_{mi} = \tau_{qi} \dot{q}_i$$

- Load torque decomposition

$$\tau_{li} = \frac{m_{ii,a}}{n_{ri}^2} \ddot{\theta}_{mi} + d_i(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$$

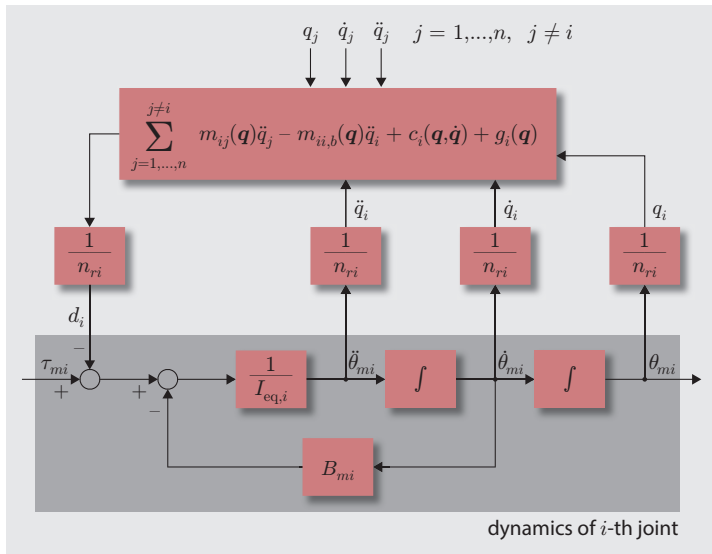
- Disturbance term

$$d_i(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \frac{1}{n_{ri}} \left( \sum_{j=1, \dots, n}^{j \neq i} m_{ij}(\mathbf{q}) \ddot{q}_j - m_{ii,b}(\mathbf{q}) \ddot{q}_i + c_i(\mathbf{q}, \dot{\mathbf{q}}) + g_i(\mathbf{q}) \right)$$

- Motor side dynamics with equivalent inertia and disturbance

$$\left( I_{mi} + \frac{m_{ii,a}}{n_{ri}^2} \right) \ddot{\theta}_{mi} + B_{mi} \dot{\theta}_{mi} = \tau_{mi} - d_i(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$$

- Block diagram of robot dynamics for independent joint control



- Regulation problem for independent joint control to be addressed using the cascaded control scheme, which can be rewritten as a PID controller
- Convergence to  $(q_{di}, 0)$  is no longer guaranteed because  $d_i$  is time-varying, yet steady state can be reached thanks to integral action (**local** result)
- Resulting controller is **decentralized**: each PID action based on the corresponding joint error
- Knowledge of system dynamic parameters is not needed, yet they affect the performance of the closed-loop system

## PD Control with Gravity Cancellation

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- Joint-space position error

$$\mathbf{e} = \mathbf{q}_d - \mathbf{q}$$

- Candidate Lyapunov function

$$V = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \mathbf{e}^T \mathbf{K}_P \mathbf{e}$$

- Time derivative of  $V$

$$\dot{V} = \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{M}}(\mathbf{q}) \dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{K}_P \mathbf{e}$$

- Using factorization  $\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}$

$$\dot{V} = \frac{1}{2} \dot{\mathbf{q}}^T \left( \dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \right) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T (\boldsymbol{\tau} - \mathbf{g}(\mathbf{q}) - \mathbf{K}_P \mathbf{e})$$

- PD control law with gravity cancellation

$$\boldsymbol{\tau} = \mathbf{K}_P \mathbf{e} - \mathbf{K}_D \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q})$$

- Resulting Lyapunov derivative

$$\dot{V} = -\dot{\mathbf{q}}^T \mathbf{K}_D \dot{\mathbf{q}}$$

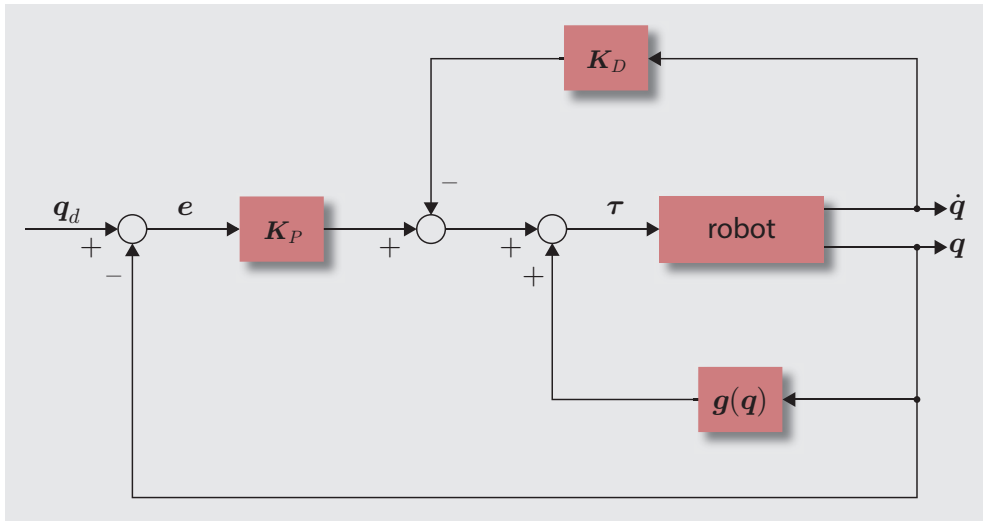
- Closed-loop relation

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{K}_P \mathbf{e}$$



- Global asymptotic stability of  $(\mathbf{q}_d, \mathbf{0})$  is guaranteed

- Block diagram of PD control with gravity cancellation



## PD Control with Gravity Compensation

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- Compensation of gravity at desired equilibrium

$$\boldsymbol{\tau} = \mathbf{K}_P \mathbf{e} - \mathbf{K}_D \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}_d)$$

- Matrix norm

$$\|\mathbf{A}\| = \sqrt{\lambda_{\max}(\mathbf{A}^T \mathbf{A})} = A_M$$

- Minimum singular value

$$A_m = \sqrt{\lambda_{\min}(\mathbf{A}^T \mathbf{A})} \geq 0$$

- Bounded Hessian condition

$$\left\| \frac{\partial^2 \mathcal{U}}{\partial \mathbf{q}^2} \right\| = \left\| \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \right\| \leq \alpha \quad \text{for all } \mathbf{q}$$

- Lipschitz bound on gravity

$$\|\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)\| \leq \alpha \|\mathbf{q} - \mathbf{q}_d\| \quad \text{for all } \mathbf{q}, \mathbf{q}_d$$

- Condition on proportional gain

$$K_{P,m} > \alpha$$

- Gain dominance inequality

$$\|\mathbf{K}_P \mathbf{e}\| \geq K_{P,m} \|\mathbf{e}\| > \alpha \|\mathbf{e}\| \geq \|\mathbf{g}(\mathbf{q}_d) - \mathbf{g}(\mathbf{q})\|$$

- Static balance relation at equilibrium

$$\mathbf{K}_P \mathbf{e} = \mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)$$

- Potential energy approximation around  $\mathbf{q}_d$

$$\mathcal{U}(\mathbf{q}) \approx \mathcal{U}(\mathbf{q}_d) + \left. \frac{\partial \mathcal{U}}{\partial \mathbf{q}} \right|_{\mathbf{q}=\mathbf{q}_d} (\mathbf{q} - \mathbf{q}_d) = \mathcal{U}(\mathbf{q}_d) - \mathbf{g}^T(\mathbf{q}_d) \mathbf{e}$$

- Candidate Lyapunov function

$$V = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \mathbf{e}^T \mathbf{K}_P \mathbf{e} + \mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) + \mathbf{e}^T \mathbf{g}(\mathbf{q}_d)$$

- Time derivative of  $V$

$$\dot{V} = \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{M}}(\mathbf{q}) \dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{K}_P \mathbf{e} + \frac{\partial \mathcal{U}}{\partial \mathbf{q}} \dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{g}(\mathbf{q}_d)$$

- Resulting  $\dot{V}$

$$\begin{aligned} \dot{V} &= \dot{\mathbf{q}}^T (\mathbf{K}_P \mathbf{e} - \mathbf{K}_D \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}_d) - \mathbf{g}(\mathbf{q})) - \dot{\mathbf{q}}^T \mathbf{K}_P \mathbf{e} + \mathbf{g}^T(\mathbf{q}) \dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{g}(\mathbf{q}_d) \\ &= -\dot{\mathbf{q}}^T \mathbf{K}_D \dot{\mathbf{q}} \end{aligned}$$

- Closed-loop dynamics

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{K}_P \mathbf{e} + \mathbf{g}(\mathbf{q}_d)$$

- Equivalent acceleration expression

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})(\mathbf{K}_P \mathbf{e} + \mathbf{g}(\mathbf{q}_d) - \mathbf{g}(\mathbf{q}))$$

$\Downarrow$

- Global asymptotic stability of  $(\mathbf{q}_d, \mathbf{0})$  is guaranteed
- Alternative scheme with integral action

$$\boldsymbol{\tau} = \mathbf{K}_P \mathbf{e} + \mathbf{K}_I \int_0^t \mathbf{e}(s) ds - \mathbf{K}_D \dot{\mathbf{q}}$$

# Trajectory Tracking in Joint Space

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- Regulation laws can track slowly time-varying references especially with high reduction ratios
- Tracking performance degrades for large velocities and accelerations due to nonlinear and coupling terms
- Model-based control uses the full robot dynamic model
- First assume perfect knowledge of the model parameters, then relax this hypothesis

## Independent PID Tracking

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- Desired joint trajectory is  $\mathbf{q}_d(t)$
- Independent controllers are designed for each joint
- Neglected inertial velocity and gravitational terms collected as  $d_i$  act as persistent time-varying disturbances
- Add a feedforward torque  $\tau_{di}(t)$  at joint  $i$ 
  - $\tau_{di}(t)$  is the  $i$ -th component of  $\boldsymbol{\tau}_d(t) = \boldsymbol{\tau}_d(\mathbf{q}_d(t), \dot{\mathbf{q}}_d(t), \ddot{\mathbf{q}}_d(t))$
- $\boldsymbol{\tau}_d(t)$  is computed along  $\mathbf{q}_d(t)$  through inverse dynamics of the robot
- The feedforward term can be scaled by  $\alpha_{mi}$  and added to the PID output
- Feedback acts locally on the trajectory error  $q_{di} - q_i$
- Feedforward is centralized since  $\tau_{di}(t)$  includes the full robot dynamics
- Tracking performance is satisfactory for smooth trajectories of limited speed and acceleration especially with large reduction gears

## Control Based on Feedback Linearization

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- Desired joint trajectory is  $\mathbf{q}_d(t)$
- Robot dynamics

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) = \boldsymbol{\tau}$$

- Nonlinear terms

$$\mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$

- Exact linearizing control law

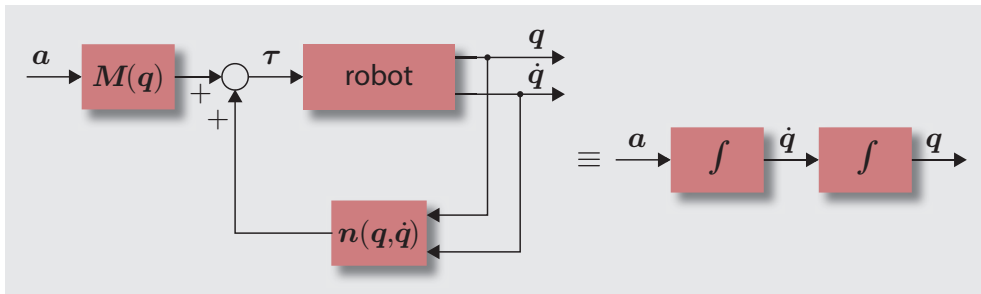
$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\mathbf{a} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

- Closed-loop dynamics

$$\ddot{\mathbf{q}} = \mathbf{a}$$

and the system becomes  $n$  decoupled chains of double integrators

- Linearizing effect of control law



- New acceleration input

$$\mathbf{a} = \ddot{\mathbf{q}}_d + \mathbf{K}_P \mathbf{e} + \mathbf{K}_D \dot{\mathbf{e}}$$

where  $\mathbf{e} = \mathbf{q}_d - \mathbf{q}$  is the tracking error

- Error dynamics

$$\ddot{\mathbf{e}} + \mathbf{K}_D \dot{\mathbf{e}} + \mathbf{K}_P \mathbf{e} = \mathbf{0}$$

- The origin is globally exponentially stable
- Control torque expression

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})(\ddot{\mathbf{q}}_d + \mathbf{K}_P \mathbf{e} + \mathbf{K}_D \dot{\mathbf{e}}) + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

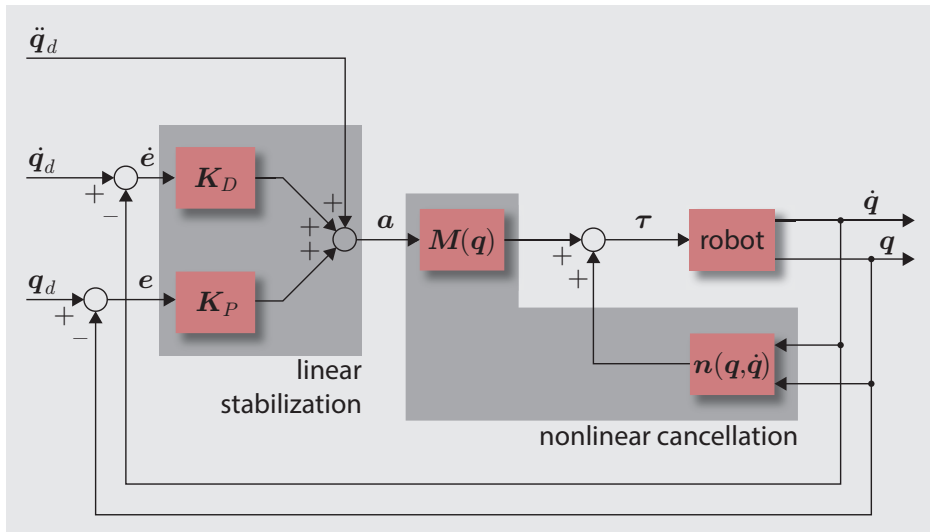
- Gain choice for independent second-order error transients uses diagonal  $\mathbf{K}_P$  and  $\mathbf{K}_D$

$$\mathbf{K}_P = \text{diag}\{\omega_{n1}^2, \dots, \omega_{nn}^2\} \quad \mathbf{K}_D = \text{diag}\{2\zeta_1\omega_{n1}, \dots, 2\zeta_n\omega_{nn}\}$$

- Regulation case with constant  $\mathbf{q}_d$

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})(\mathbf{K}_P \mathbf{e} - \mathbf{K}_D \dot{\mathbf{q}}) + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

- Block diagram of joint-space trajectory tracking via feedback linearization control



- State and input

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} \quad \mathbf{u} = \boldsymbol{\tau}$$

- Nonlinear affine dynamics

$$\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ -\mathbf{M}^{-1}(x_1)\mathbf{n}(x) \end{pmatrix} + \begin{pmatrix} \mathbf{O} \\ \mathbf{M}^{-1}(x_1) \end{pmatrix} \mathbf{u} = \mathbf{f}(x) + \mathbf{G}(x)\mathbf{u}$$

- Inverse inertia matrix notation

$$\mathbf{H}(\mathbf{x}_1) = \begin{pmatrix} \mathbf{h}_1(\mathbf{x}_1) & \dots & \mathbf{h}_n(\mathbf{x}_1) \end{pmatrix} = \mathbf{M}^{-1}(\mathbf{x}_1)$$

- First distribution

$$\Delta_{L0} = \text{span} \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{h}_1(\mathbf{x}_1) \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{0} \\ \mathbf{h}_n(\mathbf{x}_1) \end{pmatrix} \right\}$$

- Lie brackets for all  $i$  and  $j$

$$\begin{aligned} [\mathbf{g}_i, \mathbf{g}_j] &= \left[ \begin{pmatrix} \mathbf{0} \\ \mathbf{h}_i(\mathbf{x}_1) \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{h}_j(\mathbf{x}_1) \end{pmatrix} \right] \\ &= \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \frac{\partial \mathbf{h}_j}{\partial \mathbf{x}_1} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{h}_i(\mathbf{x}_1) \end{pmatrix} - \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \frac{\partial \mathbf{h}_i}{\partial \mathbf{x}_1} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{h}_j(\mathbf{x}_1) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{aligned}$$

- Computation of  $[f, g_j]$

$$[f, g_j] = \begin{pmatrix} O & O \\ \frac{\partial h_j}{\partial x_1} & O \end{pmatrix} \begin{pmatrix} x_2 \\ -H(x_1)n(x) \end{pmatrix} - \begin{pmatrix} O & I \\ * & * \end{pmatrix} \begin{pmatrix} 0 \\ h_j(x_1) \end{pmatrix} = \begin{pmatrix} h_j(x_1) \\ * \end{pmatrix}$$

- Involution distribution

$$\Delta_{L1} = \text{span} \left\{ \begin{pmatrix} 0 \\ h_1(x_1) \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ h_n(x_1) \end{pmatrix}, \begin{pmatrix} h_1(x_1) \\ * \end{pmatrix}, \dots, \begin{pmatrix} h_n(x_1) \\ * \end{pmatrix} \right\}$$

- Output selection

$$\mathbf{y} = \mathbf{x}_1 = \mathbf{q}$$

- Output derivatives

$$\dot{\mathbf{y}} = \mathbf{x}_2$$

$$\ddot{\mathbf{y}} = -\mathbf{H}(\mathbf{x}_1)\mathbf{n}(\mathbf{x}) + \mathbf{H}(\mathbf{x}_1)\mathbf{u}$$

- Input transformation

$$\mathbf{u} = \mathbf{M}(\mathbf{q})\mathbf{a} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

- Linearized input–output relation

$$\ddot{\mathbf{y}} = \mathbf{a}$$

## Control Based on Inverse Dynamics

---

- Evaluate model-based terms along  $\mathbf{q}_d(t)$  to reduce real-time computational load
- First alternative version with feedforward torque along reference trajectory (**computed torque**)

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{M}(\mathbf{q}_d)(\ddot{\mathbf{q}}_d + \mathbf{K}_P \mathbf{e} + \mathbf{K}_D \dot{\mathbf{e}}) + \mathbf{n}(\mathbf{q}_d, \dot{\mathbf{q}}_d) \\ &= \boldsymbol{\tau}_d + \mathbf{M}(\mathbf{q}_d)(\mathbf{K}_P \mathbf{e} + \mathbf{K}_D \dot{\mathbf{e}})\end{aligned}$$

- Second alternative version

$$\boldsymbol{\tau} = \boldsymbol{\tau}_d + \mathbf{K}_P \mathbf{e} + \mathbf{K}_D \dot{\mathbf{e}}$$

- Third alternative version

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}_d + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}_d + \mathbf{g}(\mathbf{q}) + \mathbf{K}_P \mathbf{e} + \mathbf{K}_D \dot{\mathbf{e}}$$

## Adaptive Control

---

- Model **uncertainty** can be **structured** or **unstructured**
- Define reference velocity and filtered error

$$\dot{q}_r = \dot{q}_d + \Sigma e$$

$$\sigma = \dot{q}_r - \dot{q} = \dot{e} + \Sigma e$$

- Controller with perfect model (**passivity-based control**)

$$\tau = M(q)\ddot{q}_r + C(q, \dot{q})\dot{q}_r + g(q) + K_D\sigma$$

- Closed-loop error dynamics

$$M(q)\dot{\sigma} + C(q, \dot{q})\sigma + K_D\sigma = 0$$

- Candidate function

$$V = \frac{1}{2}\boldsymbol{\sigma}^T \mathbf{M}(\mathbf{q})\boldsymbol{\sigma} + \frac{1}{2}\mathbf{e}^T \mathbf{Q}\mathbf{e}$$

- Time derivative

$$\dot{V} = \boldsymbol{\sigma}^T \mathbf{M}(\mathbf{q})\dot{\boldsymbol{\sigma}} + \frac{1}{2}\boldsymbol{\sigma}^T \dot{\mathbf{M}}(\mathbf{q})\boldsymbol{\sigma} + \mathbf{e}^T \mathbf{Q}\dot{\mathbf{e}} = -\boldsymbol{\sigma}^T \mathbf{K}_D \boldsymbol{\sigma} + \mathbf{e}^T \mathbf{Q}\dot{\mathbf{e}}$$

- With  $\mathbf{Q} = 2\boldsymbol{\Sigma}\mathbf{K}_D$  one obtains

$$\dot{V} = -\dot{\mathbf{e}}^T \mathbf{K}_D \dot{\mathbf{e}} - \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{K}_D \boldsymbol{\Sigma} \mathbf{e}$$

- Closed-loop system is time-varying ( $\boldsymbol{\sigma}$  depends on  $\mathbf{q}_d(t)$ )  $\Rightarrow$  Global asymptotic stability by applying Barbalat lemma:  $V$  is lower bounded by zero,  $\dot{V}$  is nonpositive, and  $\ddot{V}$  is bounded

$$\|\mathbf{M}^{-1}(\mathbf{q})\| \leq \beta_M \quad \|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\| \leq \beta_C \|\dot{\mathbf{q}}\|$$

- Robot dynamics **linear in parameters**

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\boldsymbol{\rho}$$

- Control with estimated parameters

$$\boldsymbol{\tau} = \hat{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}}_r + \hat{\mathbf{C}}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}_r + \hat{\mathbf{g}}(\mathbf{q}) + \mathbf{K}_D\boldsymbol{\sigma} = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)\hat{\boldsymbol{\rho}} + \mathbf{K}_D\boldsymbol{\sigma}$$

- Parameter error

$$\tilde{\boldsymbol{\rho}} = \boldsymbol{\rho} - \hat{\boldsymbol{\rho}}$$

- Resulting dynamics

$$\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\sigma}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\boldsymbol{\sigma} + \mathbf{K}_D\boldsymbol{\sigma} = \tilde{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}}_r + \tilde{\mathbf{C}}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}_r + \tilde{\mathbf{g}}(\mathbf{q}) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)\tilde{\boldsymbol{\rho}}$$

- Lyapunov function

$$V = \frac{1}{2} \sigma^T M(q) \sigma + \frac{1}{2} e^T Q e + \frac{1}{2} \tilde{\rho}^T K_{\rho}^{-1} \tilde{\rho}$$

- Time derivative

$$\dot{V} = -\dot{e}^T K_D \dot{e} - e^T \Sigma K_D \Sigma e + \tilde{\rho}^T (Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r) \sigma - K_{\rho}^{-1} \dot{\hat{\rho}})$$

- Adaptation law

$$\dot{\hat{\rho}} = K_{\rho} Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r) \sigma$$

- At steady state

$$Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r) (\rho - \hat{\rho}) = 0$$

- $\hat{\rho}$  is ultimately bounded but does **not** converge to  $\rho$
- convergence depends on the structure of  $Y$  (**persistence of excitation**)

- Lyapunov derivative

$$\dot{V} = -\dot{e}^T \mathbf{K}_D \dot{e} - e^T \mathbf{\Sigma} \mathbf{K}_D \mathbf{\Sigma} e$$

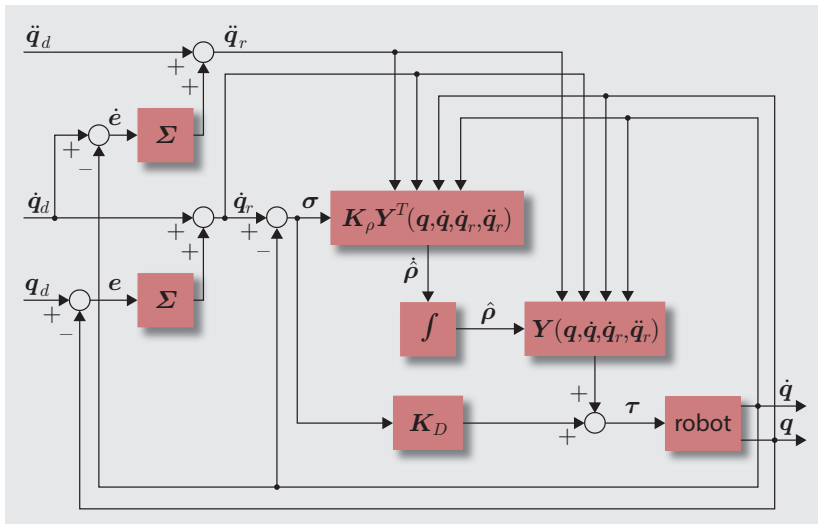
- Implemented controller form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r) \hat{\boldsymbol{\rho}} + \mathbf{K}_D(\dot{e} + \mathbf{\Sigma} e)$$

- Implemented adaptation law form

$$\dot{\hat{\boldsymbol{\rho}}} = \mathbf{K}_\rho \mathbf{Y}^T(\mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)(\dot{e} + \mathbf{\Sigma} e)$$

- Block diagram of joint-space tracking via adaptive control



- In the regulation case (constant  $q_d$ ) the adaptive control becomes

$$\begin{aligned}\tau &= -\hat{M}(q)\Sigma\dot{q} + \hat{C}(q, \dot{q})\Sigma e + \hat{g}(q) + K_D (\Sigma e - \dot{q}) \\ &= Y(q, \dot{q}, \Sigma e, -\Sigma\dot{q})\hat{\rho} + K_D (\Sigma e - \dot{q}) \\ \dot{\hat{\rho}} &= K_\rho Y^T(q, \dot{q}, \Sigma e, -\Sigma\dot{q}) (\Sigma e - \dot{q})\end{aligned}$$

- In the presence of unmodeled dynamics (simplified model or external disturbances) performance of adaptive control is degraded

## Robust Control

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- Uncertainty on dynamic coefficients

$$\|\tilde{\rho}\| \leq \bar{\rho}$$

- Robust control law

$$\tau = Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r) (\hat{\rho} + u) + K_D \sigma$$

- Define

$$z = Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r) \sigma$$

- Lyapunov derivative term

$$\dot{V} = -\dot{e}^T K_D \dot{e} - e^T \Sigma K_D \Sigma e - z^T (\tilde{\rho} + u)$$

- Discontinuous robust control law

$$\mathbf{u} = \begin{cases} \frac{\bar{\rho}}{\|\mathbf{z}\|} \mathbf{z} & \text{for } \|\mathbf{z}\| \neq 0 \\ \mathbf{0} & \text{for } \|\mathbf{z}\| = 0 \end{cases}$$

- Component-wise bounded uncertainty

$$|\tilde{\rho}_i| \leq \bar{\rho}_i \quad i = 1, \dots, r$$

- Inequality

$$\begin{aligned} -\mathbf{z}^T(\tilde{\boldsymbol{\rho}} + \mathbf{u}) &= -\mathbf{z}^T \tilde{\boldsymbol{\rho}} - \frac{\bar{\rho}}{\|\mathbf{z}\|} \mathbf{z}^T \mathbf{z} \\ &\leq \|\mathbf{z}\| \|\tilde{\boldsymbol{\rho}}\| - \bar{\rho} \|\mathbf{z}\| = -\|\mathbf{z}\| (\bar{\rho} - \|\tilde{\boldsymbol{\rho}}\|) \leq 0 \end{aligned}$$

- Continuous approximation with boundary layer  $\epsilon$

$$\mathbf{u} = \begin{cases} \frac{\bar{\rho}}{\|\mathbf{z}\|} \mathbf{z} & \text{for } \|\mathbf{z}\| > \epsilon \\ \frac{\bar{\rho}}{\epsilon} \mathbf{z} & \text{for } \|\mathbf{z}\| \leq \epsilon \end{cases}$$

- Resulting bound

$$-\mathbf{z}^T(\tilde{\boldsymbol{\rho}} + \mathbf{u}) = -\mathbf{z}^T\tilde{\boldsymbol{\rho}} - \frac{\bar{\rho}}{\epsilon} \mathbf{z}^T \mathbf{z} \leq \bar{\rho} \|\mathbf{z}\| - \frac{\bar{\rho}}{\epsilon} \|\mathbf{z}\|^2$$

- When  $\|\mathbf{z}\| = \epsilon/2$ , the bound attains its maximum value  $\epsilon \bar{\rho}/4$

$\Downarrow$

- Lyapunov bound

$$\dot{V} \leq -\dot{\mathbf{e}}^T \mathbf{K}_D \dot{\mathbf{e}} - \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{K}_D \boldsymbol{\Sigma} \mathbf{e} + \frac{1}{4} \epsilon \bar{\rho}$$

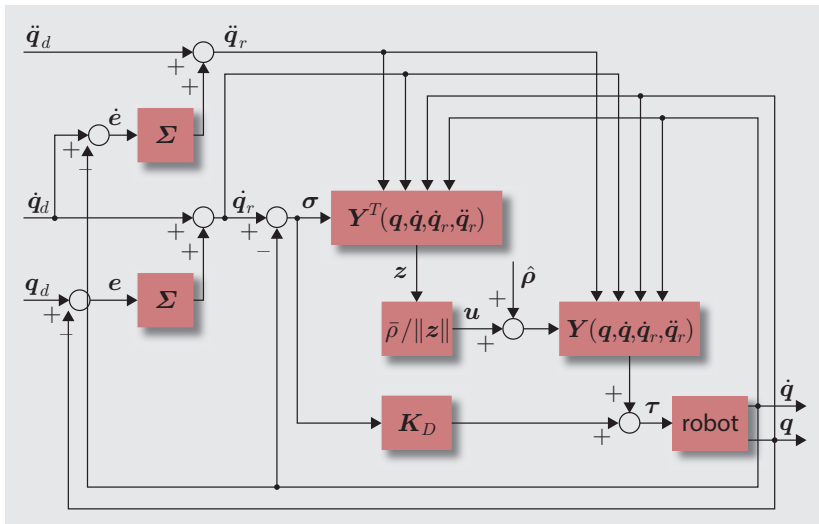
- $\dot{V}$  is negative outside a region  $M$  containing the equilibrium, whose extent depends on the product of the uncertainty bound  $\bar{\rho}$  by the constant  $\epsilon \Rightarrow e$  will remain bounded and the evolution of the closed-loop system converges to  $M$  in finite time; the size of  $M$  can be tuned by choosing  $\epsilon$
- Different uncertainty bound for each coefficient

$$|\tilde{\rho}_i| \leq \bar{\rho}_i \quad i = 1, \dots, r$$

- Component-wise law

$$u_i = \begin{cases} \frac{\bar{\rho}_i}{|z_i|} z_i & \text{for } |z_i| \geq \epsilon_i \\ \frac{\bar{\rho}_i}{\epsilon_i} z_i & \text{for } |z_i| < \epsilon_i \end{cases}$$

- Block diagram of joint-space tracking via robust control



# Control in Task Space

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- Previous schemes assume desired motion specified in joint space or converted to joint space
- Task-space schemes operate directly in task space
- Task-space schemes are computationally more demanding
- Task-space schemes are useful when the task is updated in real time using exteroceptive sensors

## PD Control with Gravity Cancellation

---

- Regulate task variables  $\mathbf{y}$  to a desired constant value  $\mathbf{y}_d$
- Define the task error

$$\mathbf{e}_y = \mathbf{y}_d - \mathbf{y}$$

- Define the Lyapunov candidate

$$V = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \mathbf{e}_y^T \mathbf{K}_P \mathbf{e}_y$$

- Time derivative of the Lyapunov candidate

$$\dot{V} = \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{M}}(\mathbf{q}) \dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{J}^T(\mathbf{q}) \mathbf{K}_P \mathbf{e}_y$$

- Equivalent expression

$$\dot{V} = \dot{\mathbf{q}}^T (\boldsymbol{\tau} - \mathbf{g}(\mathbf{q}) - \mathbf{J}^T(\mathbf{q}) \mathbf{K}_P \mathbf{e}_y)$$

- PD control law with gravity cancellation

$$\boldsymbol{\tau} = \mathbf{J}^T(\mathbf{q})\mathbf{K}_P\mathbf{e}_y - \mathbf{K}_D\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q})$$

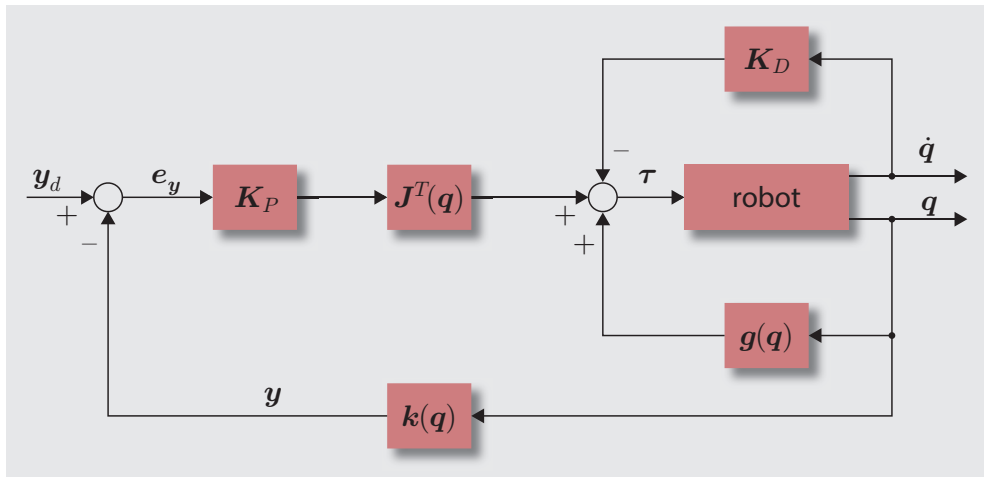
- Resulting Lyapunov derivative

$$\dot{V} = -\dot{\mathbf{q}}^T \mathbf{K}_D \dot{\mathbf{q}}$$

- Alternative form

$$\boldsymbol{\tau} = \mathbf{J}^T(\mathbf{q})(\mathbf{K}_P\mathbf{e}_y - \mathbf{K}_D\mathbf{J}(\mathbf{q})\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q})$$

- Block diagram of task-space regulation via PD control with gravity cancellation



- Task error partition

$$\mathbf{e}_y = \begin{pmatrix} \mathbf{e}_p \\ \mathbf{e}_o \end{pmatrix} = \begin{pmatrix} \mathbf{p}_d - \mathbf{p}_e \\ \phi_{ed} \end{pmatrix}$$

- Task error derivative

$$\dot{\mathbf{e}}_y = -\mathbf{J}_e(\mathbf{q})\dot{\mathbf{q}}$$

- End-effector analytic Jacobian

$$\mathbf{J}_e(\mathbf{q}) = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{T}_E^{-1}(\phi_{ed}) \end{pmatrix} \mathbf{J}_g(\mathbf{q})$$

## Control Based on Feedback Linearization

---

- Use the task acceleration relation

$$\ddot{\mathbf{y}} = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{J}(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})(\boldsymbol{\tau} - \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})) + \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}}$$

- Choose the control torque

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\mathbf{J}^{-1}(\mathbf{q})(\mathbf{a}_y - \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}}) + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

- Obtain the linear relation

$$\ddot{\mathbf{y}} = \mathbf{a}_y$$

- Choose the task-space acceleration input

$$\mathbf{a}_y = \ddot{\mathbf{y}}_d + \mathbf{K}_P \mathbf{e}_y + \mathbf{K}_D \dot{\mathbf{e}}_y$$

- Obtain the error dynamics

$$\ddot{\mathbf{e}}_y = \ddot{\mathbf{y}}_d - \ddot{\mathbf{y}} = \ddot{\mathbf{y}}_d - \mathbf{a}_y = -\mathbf{K}_P \mathbf{e}_y - \mathbf{K}_D \dot{\mathbf{e}}_y$$

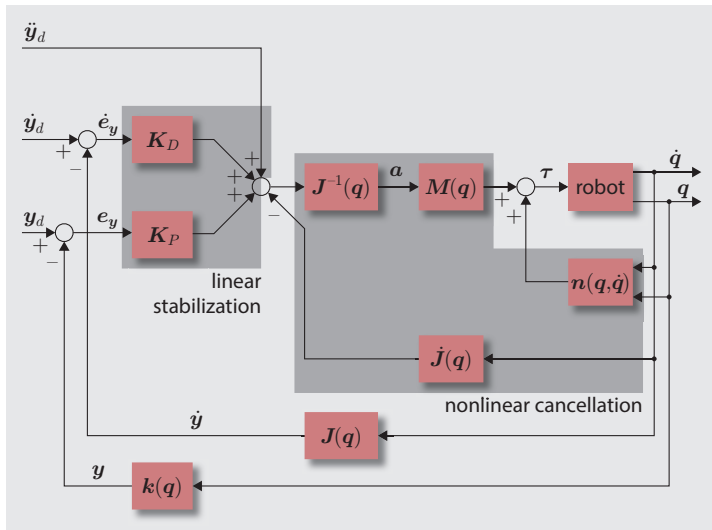
- Final torque expression

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\mathbf{J}^{-1}(\mathbf{q})(\ddot{\mathbf{y}}_d + \mathbf{K}_P \mathbf{e}_y + \mathbf{K}_D \dot{\mathbf{e}}_y - \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}}) + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

- Regulation variant

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\mathbf{J}^{-1}(\mathbf{q})(\mathbf{K}_P \mathbf{e}_y - \mathbf{K}_D \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} - \dot{\mathbf{J}}(\mathbf{q})\dot{\mathbf{q}}) + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

- Block diagram of task-space trajectory tracking via feedback linearization control



- Task-space dynamic model

$$\mathbf{\Lambda}(\mathbf{q})\ddot{\mathbf{y}} + \boldsymbol{\psi}(\mathbf{q}, \dot{\mathbf{q}}) + \boldsymbol{\eta}(\mathbf{q}) = \boldsymbol{\gamma}$$

- Equivalent expression

$$\boldsymbol{\gamma} = \mathbf{\Lambda}(\mathbf{q})\mathbf{a}_y + \boldsymbol{\psi}(\mathbf{q}, \dot{\mathbf{q}}) + \boldsymbol{\eta}(\mathbf{q})$$

- Tracking expression

$$\boldsymbol{\gamma} = \mathbf{\Lambda}(\mathbf{q})(\ddot{\mathbf{y}}_d + \mathbf{K}_P\mathbf{e}_y + \mathbf{K}_D\dot{\mathbf{e}}_y) + \boldsymbol{\psi}(\mathbf{q}, \dot{\mathbf{q}}) + \boldsymbol{\eta}(\mathbf{q})$$

- End-effector twist derivative

$$\dot{\mathbf{t}}_e = \begin{pmatrix} \dot{\mathbf{v}}_e \\ \dot{\boldsymbol{\omega}}_e \end{pmatrix} = \mathbf{J}_g(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}_g(\mathbf{q})\dot{\mathbf{q}}$$

- Alternative expression

$$\dot{\mathbf{t}}_e = \mathbf{J}_g(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})(\boldsymbol{\tau} - \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})) + \dot{\mathbf{J}}_g(\mathbf{q})\dot{\mathbf{q}}$$

- Control torque

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\mathbf{J}_g^{-1}(\mathbf{q})(\mathbf{a}_e - \dot{\mathbf{J}}_g(\mathbf{q})\dot{\mathbf{q}}) + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

- Linear relation

$$\dot{\mathbf{t}}_e = \mathbf{a}_e \quad \text{or} \quad \begin{pmatrix} \dot{\mathbf{v}}_e \\ \dot{\boldsymbol{\omega}}_e \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{e,p} \\ \mathbf{a}_{e,o} \end{pmatrix}$$

- Position component

$$\mathbf{a}_{e,p} = \dot{\mathbf{v}}_d + \mathbf{K}_{P,p}\mathbf{e}_p + \mathbf{K}_{D,p}\dot{\mathbf{e}}_p$$

- Position error dynamics

$$\ddot{\mathbf{e}}_p = -\mathbf{K}_{P,p}\mathbf{e}_p - \mathbf{K}_{D,p}\dot{\mathbf{e}}_p$$

- Angular velocity relation

$$\dot{\boldsymbol{\omega}}_e = \dot{\boldsymbol{\omega}}_d - \mathbf{T}_E(\phi_{ed})\ddot{\phi}_{ed} - \dot{\mathbf{T}}_E(\phi_{ed})\dot{\phi}_{ed}$$

- Orientation component

$$\mathbf{a}_{e,o} = \dot{\boldsymbol{\omega}}_d - \dot{\mathbf{T}}_E(\phi_{ed})\dot{\phi}_{ed} + \mathbf{T}_E(\phi_{ed})(\mathbf{K}_{P,o}\phi_{ed} + \mathbf{K}_{D,o}\dot{\phi}_{ed})$$

- Orientation error dynamics

$$\ddot{\phi}_{ed} = -\mathbf{K}_{P,o}\phi_{ed} - \mathbf{K}_{D,o}\dot{\phi}_{ed}$$

## The Case of Redundant Robots

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- Joint-space dynamic model and direct kinematics

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})(\boldsymbol{\tau} - \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}))$$
$$\mathbf{y} = \mathbf{k}(\mathbf{q})$$

- Task acceleration relation

$$\ddot{\mathbf{y}} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}$$

- General torque expression

$$\boldsymbol{\tau} = (\boldsymbol{J}\boldsymbol{M}^{-1})_{\boldsymbol{W}}^{\dagger}(\boldsymbol{a}_y - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}) + (\boldsymbol{I} - (\boldsymbol{J}\boldsymbol{M}^{-1})_{\boldsymbol{W}}^{\dagger}\boldsymbol{J}\boldsymbol{M}^{-1})\boldsymbol{\tau}_N + \boldsymbol{n}$$

- Weighted pseudoinverse expression ( $\boldsymbol{W} = \boldsymbol{M}^{-1}$ )

$$(\boldsymbol{J}\boldsymbol{M}^{-1})_{\boldsymbol{M}^{-1}}^{\dagger} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1} = \boldsymbol{J}^T\boldsymbol{\Lambda}$$

- Null-space projector

$$\boldsymbol{I} - (\boldsymbol{J}\boldsymbol{M}^{-1})_{\boldsymbol{M}^{-1}}^{\dagger}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{I} - \boldsymbol{J}^T\boldsymbol{\Lambda}\boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{J}_M^{\dagger})^T$$

- Equivalent form

$$\boldsymbol{\tau} = \mathbf{J}^T \boldsymbol{\Lambda} (\mathbf{a}_y - \dot{\mathbf{J}} \dot{\mathbf{q}}) + (\mathbf{I} - \mathbf{J}_M^\dagger \mathbf{J})^T \boldsymbol{\tau}_N + \mathbf{n}$$

- Decomposition identity

$$\mathbf{n} = \mathbf{n} + \mathbf{J}^T (\mathbf{J}_M^\dagger)^T \mathbf{n} - \mathbf{J}^T (\mathbf{J}_M^\dagger)^T \mathbf{n} = \mathbf{J}^T (\mathbf{J}_M^\dagger)^T \mathbf{n} + (\mathbf{I} - \mathbf{J}_M^\dagger \mathbf{J})^T \mathbf{n}$$

- Torque expression

$$\boldsymbol{\tau} = \mathbf{J}^T (\boldsymbol{\Lambda} (\mathbf{a}_y - \dot{\mathbf{J}} \dot{\mathbf{q}}) + (\mathbf{J}_M^\dagger)^T \mathbf{n}) + (\mathbf{I} - \mathbf{J}_M^\dagger \mathbf{J})^T (\boldsymbol{\tau}_N + \mathbf{n})$$

- Alternative pseudoinverse with  $\mathbf{W} = \mathbf{M}^{-2}$

$$(\mathbf{J}\mathbf{M}^{-1})_{\mathbf{M}^{-2}}^{\dagger} = \mathbf{M}\mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1} = \mathbf{M}\mathbf{J}^{\dagger}$$

- Associated projector

$$\mathbf{I} - (\mathbf{J}\mathbf{M}^{-1})_{\mathbf{M}^{-2}}^{\dagger}\mathbf{J}\mathbf{M}^{-1} = \mathbf{I} - \mathbf{M}\mathbf{J}^{\dagger}\mathbf{J}\mathbf{M}^{-1} = \mathbf{M}(\mathbf{I} - \mathbf{J}^{\dagger}\mathbf{J})\mathbf{M}^{-1}$$

- Resulting torque ( $\mathbf{M}$  not involved in pseudoinversion)

$$\boldsymbol{\tau} = \mathbf{M}\mathbf{J}^{\dagger}(\mathbf{a}_y - \dot{\mathbf{J}}\dot{\mathbf{q}}) + \mathbf{M}(\mathbf{I} - \mathbf{J}^{\dagger}\mathbf{J})\mathbf{M}^{-1}\boldsymbol{\tau}_N + \mathbf{n}$$

- Acceleration decomposition

$$\ddot{\mathbf{q}} = \mathbf{J}_V^\dagger (\mathbf{a}_y - \dot{\mathbf{J}}\dot{\mathbf{q}}) + (\mathbf{I} - \mathbf{J}_V^\dagger \mathbf{J}) \ddot{\mathbf{q}}_N$$

- Resulting torque

$$\boldsymbol{\tau} = \mathbf{M} \mathbf{J}_V^\dagger (\mathbf{a}_y - \dot{\mathbf{J}}\dot{\mathbf{q}}) + \mathbf{M} (\mathbf{I} - \mathbf{J}_V^\dagger \mathbf{J}) \ddot{\mathbf{q}}_N + \mathbf{n}$$

- Relation with  $\mathbf{V} = \mathbf{M}$

$$\mathbf{M} \mathbf{J}_M^\dagger = \mathbf{M} \mathbf{M}^{-1} \mathbf{J}^T (\mathbf{J} \mathbf{M}^{-1} \mathbf{J}^T)^{-1} = \mathbf{J}^T \boldsymbol{\Lambda}$$

- Resulting torque with  $\ddot{\mathbf{q}}_N = \mathbf{M}^{-1} \boldsymbol{\tau}_N$

$$\boldsymbol{\tau} = \mathbf{J}^T \boldsymbol{\Lambda} (\mathbf{a}_y - \dot{\mathbf{J}}\dot{\mathbf{q}}) + \mathbf{M} (\mathbf{I} - \mathbf{J}_M^\dagger \mathbf{J}) \mathbf{M}^{-1} \boldsymbol{\tau}_N + \mathbf{n}$$

- Identical to previous linearizing torque  $\boldsymbol{\tau} = \mathbf{J}^T \boldsymbol{\Lambda} (\mathbf{a}_y - \dot{\mathbf{J}}\dot{\mathbf{q}}) + (\mathbf{I} - \mathbf{J}_M^\dagger \mathbf{J})^T \boldsymbol{\tau}_N + \mathbf{n}$  since

$$\mathbf{M} (\mathbf{I} - \mathbf{J}_M^\dagger \mathbf{J}) \mathbf{M}^{-1} = \mathbf{I} - \mathbf{M} \mathbf{J}_M^\dagger \mathbf{J} \mathbf{M}^{-1} = (\mathbf{I} - \mathbf{J}_M^\dagger \mathbf{J})^T$$